Chapter 2

Vector analysis in special relativity

2.1 Definition of a vector

2.2 Vector algebra

2.3 The four-velocity

An object's four velocity, denoted $\vec{U}$, is the vector tangent to its world line, with unit length. This means it extends one unit in time, and zero in space, so it is timelike.

For an *accelerated* particle (which we have not considered up to now), we may not be able to define an inertial frame, but we *can* define a **momentarily comoving reference frame** (MCRF) which, as the name suggests, moves with the same velocity as the observer for an infinitesimal period of time. We can therefore construct a continuous sequence of MCRFs for any object. If an object has MCRF $\mathcal{O}$, then its four-velocity is *defined* to be the basis vector $\vec{e}_0$.

2.4 The four-momentum

Analogous to the three-momentum, we define the four-momentum to be

$$\vec{p} = m\vec{U}. \qquad \text{(Schutz 2.19)}$$

It has components

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (E, p^1, p^2, p^3). \qquad \text{(Schutz 2.20)}$$

Calling p^0 "E" is no accident, it is in fact the energy. There is an interesting consequence to this: since vectors are invariant with respect to reference frame, but vector components are not, this means that the four-momentum does not change in different reference frames, but the energy *does*. One example would be the doppler effect, which causes the color (or energy) of a photon to shift depending on the radial velocity of the source and observer.

2.5 Scalar product

$$\vec{A} \cdot \vec{B} = -(A^0 B^0) + (A^1 B^1) + (A^2 B^2) + (A^3 B^3)$$

2.6 Applications

2.7 Photons

$\vec{x} \cdot \vec{x} = 0$, so we cannot define $\vec{U}$ for photons. We can, however, define $\vec{p}$. Since $\vec{p} \cdot \vec{p} = -m^2$, and photons are massless, we have $\vec{p} \cdot \vec{p} = 0$.

2.8 Further reading

2.9 Exercises

2 Identify the free and dummy indices in the following equations, and write equivalent expressions with different indices. Also, write how many equations are represented by each expression.

Note, I will express the set of free indices by $\mathcal{F}$ and the set of dummy indices as $\mathcal{D}$, and I will use the original index names.

(a) $A^\alpha B_\beta = 5 \implies A^\beta B_\alpha = 5$ (16 equations, $\mathcal{F} = \{\alpha, \beta\}$, $\mathcal{D} = \emptyset$)

(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\nu A^\nu \implies A^{\bar{\nu}} = \Lambda^{\bar{\nu}}{}_\mu A^\mu$ (4 equations, $\mathcal{F} = \{\bar{\mu}\}$, $\mathcal{D} = \{\nu\}$).

(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha} \implies T^{\eta\phi\theta} A_\phi C_\theta{}^\zeta = D^{\zeta\eta}$ (16 equations, $\mathcal{F} = \{\alpha, \gamma\}$, $\mathcal{D} = \{\mu, \lambda\}$)

(d) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = G_{\mu\nu} \implies R_{\chi\epsilon} - \frac{1}{2} g_{\chi\epsilon} = G_{\chi\epsilon}$ (16 equations, $\mathcal{F} = \{\mu, \nu\}$, $\mathcal{D} = \emptyset$)

4 Given vectors $\vec{A} \to_{\mathcal{O}} (5, -1, 0, 1)$ and $\vec{B} \to_{\mathcal{O}} (-2, 1, 1, -6)$, find the components in $\mathcal{O}$ of

(a) $-6\vec{A} \to_{\mathcal{O}} (-30, 6, 0, -6)$

(b) $3\vec{A} + \vec{B} \to_{\mathcal{O}} (13, -2, 1, -3)$

(c) $-6\vec{A} + 3\vec{B} \to_{\mathcal{O}} (-36, 9, 3, -24)$

6 Draw a spacetime diagram from $\mathcal{O}$'s reference frame. There are two other frames, $\bar{\mathcal{O}}$ and $\bar{\bar{\mathcal{O}}}$, which are each moving with velocity 0.6 in the $+x$ direction from each respective frame. Plot each frame's basis vectors, as observed by $\mathcal{O}$.

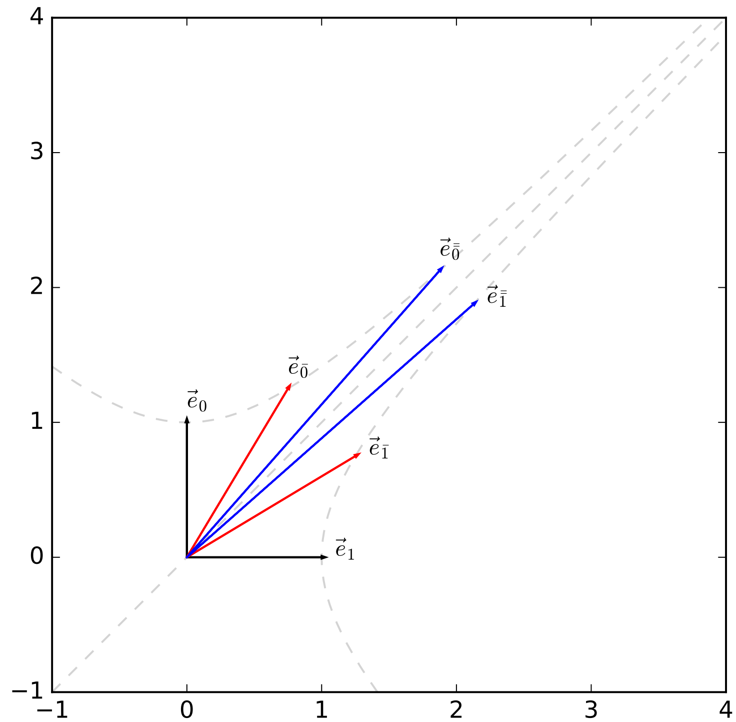

Figure 2.1: Exercise 6

See Figure 2.1.

9 Prove, by writing out all the terms that

$$\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)=\sum_{\beta=0}^{3}\left(\sum_{\bar{\alpha}=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)$$

$$\begin{aligned}
\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right) &= \sum_{\bar{\alpha}=0}^{3}\left(\Lambda^{\bar{\alpha}}{}_{0}A^{0}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{1}A^{1}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{2}A^{2}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{3}A^{3}\vec{e}_{\bar{\alpha}}\right)\\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{3}A^{3}\vec{e}_{\bar{0}}\\
&+ \Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{3}A^{3}\vec{e}_{\bar{1}}\\
&+ \Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{3}A^{3}\vec{e}_{\bar{2}}\\
&+ \Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{3}A^{3}\vec{e}_{\bar{3}}\\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}}\\
&+ \Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}}\\
&+ \Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}}
\end{aligned}$$

$$
\begin{aligned}
&+ \Lambda^{\bar{0}}{}_3 A^3 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_3 A^3 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_3 A^3 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_3 A^3 \vec{e}_{\bar{3}} \\
&= \sum_{\beta=0}^{3} \left(\Lambda^{\bar{0}}{}_\beta A^\beta \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_\beta A^\beta \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_\beta A^\beta \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_\beta A^\beta \vec{e}_{\bar{3}} \right) \\
&= \sum_{\beta=0}^{3} \left(\sum_{\bar{\alpha}=0}^{3} \Lambda^{\bar{\alpha}}{}_\beta A^\beta \vec{e}_{\bar{\alpha}} \right)
\end{aligned}
$$

11 Let $\Lambda^{\bar{\alpha}}{}_\beta$ be the matrix of the Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$, given in Equation 1.12. Let $\vec{A}$ be an arbitrary vector with components (A^0, A^1, A^2, A^3) in frame $\mathcal{O}$.

(a) Write down the matrix of $\Lambda^{\nu}{}_{\bar{\mu}}(-v)$.

Intuitively, it should appear the same as $\Lambda^{\bar{\alpha}}{}_\beta$, but with the negative signs removed. More rigorously, it is given by the matrix inverse of $\Lambda^{\bar{\alpha}}{}_\beta$, as their product should be the identity matrix. I have used a computer algebra system (Wolfram Alpha) to take the inverse of this matrix symbolically, confirming my suspicion:

$$
\Lambda^{\nu}{}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

(b) Find $A^{\bar{\alpha}}$ for all $\bar{\alpha}$.

$$
\begin{aligned}
A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}{}_\beta A^\beta \\
A^{\bar{0}} &= \gamma(A^0 - vA^1) \\
A^{\bar{1}} &= \gamma(A^1 - vA^0) \\
A^{\bar{2}} &= A^2 \\
A^{\bar{3}} &= A^3
\end{aligned}
$$

(c) Verify Equation 2.18 by performing the sum for all values of ν and α.

To simplify things, I do this via matrix multiplication

$$
\Lambda^{\bar{\alpha}}{}_\beta(v)\Lambda^{\nu}{}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma^2 - v^2\gamma^2 & v\gamma^2 - v\gamma^2 & 0 & 0 \\ v\gamma^2 - v\gamma^2 & \gamma^2 - v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$= \begin{pmatrix} \gamma^2(1-v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\nu}{}_{\alpha}$$

(d) Write down the Lorentz transformation matrix from $\bar{\mathcal{O}}$ to $\mathcal{O}$, justifying each term.

It should just be $\Lambda^{\nu}{}_{\bar{\mu}}(-v)$. I'm not sure what else to say at this point.

(e) Using the result from part (d), find A^{β} from $A^{\bar{\alpha}}$. How does this relate to Equation 2.18?

$$\Lambda^{\beta}{}_{\bar{\alpha}} A^{\bar{\alpha}} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(A^0 - vA^1) \\ \gamma(A^1 - vA^0) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma^2(A^0 - vA^1) + v\gamma^2(A^1 - vA^0) + 0 + 0 \\ v\gamma^2(A^0 - vA^1) + \gamma^2(A^1 - vA^0) + 0 + 0 \\ A^2 \\ A^3 \end{pmatrix}$$

$$= \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) + A^1(v\gamma^2 - v\gamma^2) \\ A^0(v\gamma^2 - v^2\gamma^2) + A^1(\gamma^2 - v\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) \\ A^1(\gamma^2 - v^2\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A^{\beta}$$

Since $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}(v)$, this goes to show that $\Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v)A^{\lambda} = A^{\lambda} \implies \Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v) = \delta^{\nu}{}_{\alpha}$.

(f) Verify in the same manner as (c) that

$$\Lambda^{\nu}{}_{\bar{\beta}}(v)\Lambda^{\bar{\alpha}}{}_{\nu}(-v) = \delta^{\bar{\alpha}}{}_{\bar{\beta}}$$

My matrix multiplication approach will just give me the same result as before. Perhaps another approach was intended?

(g) Establish that

$$\vec{e}_{\alpha} = \Lambda^{\bar{\beta}}{}_{\alpha}\vec{e}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}\Lambda^{\nu}{}_{\bar{\beta}}\vec{e}_{\nu} = \delta^{\nu}{}_{\alpha}\vec{e}_{\nu}$$

$$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}A^{\alpha} = \Lambda^{\bar{\beta}}{}_{\alpha}\Lambda^{\alpha}{}_{\bar{\mu}}A^{\bar{\mu}} = \delta^{\bar{\beta}}{}_{\bar{\mu}}A^{\bar{\mu}}$$

14 The following matrix gives a Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$?

This would correspond to a Lorentz boost along the z-axis, meaning

$$\Lambda^{\bar{\alpha}}{}_{\beta}(v) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

and thus we have $\gamma = 1.25$ and $-v\gamma = 0.75$. Solving for v, we get

$$-v\gamma = \frac{3}{4} \implies v = -\frac{3}{4\gamma} = -\frac{3 \cdot 4}{4 \cdot 5} = -\frac{3}{5}.$$

So $\bar{\mathcal{O}}$ is moving with speed 0.6 relative to the $-z$-axis of $\mathcal{O}$.

(b) What is the inverse matrix to the given one?

Numerically, it comes out to be

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix},$$

which makes sense, when you consider that the inverse matrix should be a Lorentz transformation with the velocity negated.

(c) Find the components in $\mathcal{O}$ of $\vec{A} \to_{\bar{\mathcal{O}}} (1, 2, 0, 0)$.

$$\vec{A} \underset{\mathcal{O}}{\to} \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2 \\ 0 \\ -0.75 \end{pmatrix}$$

15

(a) Compute the four-velocity components in $\mathcal{O}$ of a particle whose speed is v in the $+x$-direction relative to $\mathcal{O}$, using the Lorentz transformation.

$$\begin{aligned}
\vec{U} &= \vec{e}_{\bar{0}} \\
U^\alpha &= \Lambda^\alpha{}_{\bar{\beta}}(\vec{e}_{\bar{0}})^{\bar{\beta}} = \Lambda^\alpha_{\bar{0}}, \\
U^0 &= \gamma \\
U^1 &= v\gamma \\
U^2 &= U^3 = 0
\end{aligned}$$

(b) Generalize to arbitrary velocities $\boldsymbol{v}$, where $|v| < 1$.

$$\Lambda^\alpha{}_{\bar{\beta}}(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

$$U^0 = \gamma \quad U^1 = \gamma v_x \quad U^2 = \gamma v_y \quad U^3 = \gamma v_z$$

(c) Use this result to express $\boldsymbol{v}$ as a function of the components $\{U^\alpha\}$.

$$\begin{aligned}
\boldsymbol{v} &= v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3 \\
v_i &= \frac{U^i}{\gamma} \\
\boldsymbol{v} &= \frac{1}{\gamma} U^i \vec{e}_i
\end{aligned}$$

(d) Find the three-velocity $\boldsymbol{v}$ of a particle with four-velocity components $(2, 1, 1, 1)$.

$U^0 = \gamma = 2$, and $U^i = 1$, so

$$\boldsymbol{v} = \frac{1}{2}\vec{e}_i$$

17

Not sure how to approach this problem.

(a) Prove that any timelike vector $\vec{U}$ for which $U^0 > 0$ and $\vec{U} \cdot \vec{U} = -1$ is the four-velocity of *some* world line.

(b) Use this to prove that for any timelike vector $\vec{V}$ there is a Lorentz frame in which the $\vec{V}$ has zero spatial components.

19 A body is uniformly accelerated if the four-vector $\vec{a}$ has constant spatial direction and magnitude, $\vec{a} \cdot \vec{a} = \alpha^2 \geqslant 0$.

(a) Show that this implies the components of $\vec{a}$ in the body's MCRF are all constant, and that these are equivalent to the Galilean "acceleration".

We normalize the vector $\vec{a}$ by dividing each of its terms by the magnitude of the vector, so

$$\frac{a^\lambda}{\alpha}.$$

Since α is constant, and also the *direction* is constant, this means that the above expression is *also* constant, as the normalized components tell you about the direction. If we multiply a constant by a constant, we should still get a constant, so we multiply the above expression by α, getting a^λ to be constant.

In the MCRF of an object, $\mathrm{d}\tau = \mathrm{d}t$, and so we can write

$$\vec{a} = \frac{\mathrm{d}\vec{U}}{\mathrm{d}t} = \left(0, \frac{\mathrm{d}U^1}{\mathrm{d}t}, \frac{\mathrm{d}U^2}{\mathrm{d}t}, \frac{\mathrm{d}U^3}{\mathrm{d}t}\right),$$

which is analogous to the Galilean acceleration.

(b) A body is uniformly accelerated with $\alpha = 10\,\mathrm{m/s^2}$. It starts from rest, and falls for a time t. Find its speed as a function of t, and find the time to reach $v = 0.999$.

$$\begin{aligned}
\vec{U} &\underset{\text{MCRF}}{\rightarrow} (1, 0, 0, 0) \\
&\underset{\mathcal{O}}{\rightarrow} (\gamma, \gamma v, 0, 0) \\
\frac{\mathrm{d}\vec{U}}{\mathrm{d}\tau} &\underset{\text{MCRF}}{\rightarrow} (0, \alpha, 0, 0) \\
&\underset{\mathcal{O}}{\rightarrow} (\gamma, \gamma\alpha, 0, 0) \\
U^x &= \int_0^t \frac{\mathrm{d}U^x}{\mathrm{d}\tau}\mathrm{d}\tau = \int_0^t \gamma\alpha\frac{\mathrm{d}t}{\gamma} = \int_0^t \alpha\,\mathrm{d}t = \alpha t \\
&= \gamma v = \frac{v}{\sqrt{1-v^2}} \\
v^2 &= (\alpha t)^2(1-v^2) = (\alpha t)^2 - (\alpha t v)^2 \\
v^2(1+(\alpha t)^2) &= (\alpha t)^2 \\
v^2 &= \frac{(\alpha t)^2}{1+(\alpha t)^2} \implies v = \sqrt{\frac{(\alpha t)^2}{1+(\alpha t)^2}}
\end{aligned}$$

To find the time to reach $v = 0.999$, we go back to the expression $\gamma v = \alpha t$, solve for t, and substitute for v and α. Note that in natural units, $\alpha = 10\,\mathrm{m/s^2}c^{-2} \approx 1.11 \times 10^{-16}\,\mathrm{m^{-1}}$

$$t = \frac{v}{\alpha\sqrt{1-v^2}} = \frac{0.999}{1.11 \times 10^{-16}\,\mathrm{m^{-1}}\sqrt{1-0.999^2}} \approx 2.01 \times 10^{17}\,\mathrm{m}.$$

24 Show that a positron and electron cannot annihilate to form a single photon, but they can annihilate to form two photons.

We consider the center of momentum frame, where $\sum \vec{p}_{(i)} \rightarrow_{\text{CM}} (E_{\text{total}}, 0, 0, 0)$. Without loss of generality, we assume that the velocities of the two particles are equal and opposite, such that

$$\vec{p}_{e^+} \to_{\text{CM}} m_e(\gamma, \gamma v, 0, 0), \qquad \vec{p}_{e^-} \to_{\text{CM}} m_e(\gamma, -\gamma v, 0, 0).$$

The photon they create will have to have a momentum of $\vec{p}_{\gamma,\text{single}} \to_{\text{CM}} (h\nu, h\nu, 0, 0)$. By conservation of four-momentum, we have

$$\begin{aligned} \vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,\text{single}} \\ (\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= \vec{p}_{\gamma,\text{single}} \cdot \vec{p}_{\gamma,\text{single}} \\ (\vec{p}_{e^+} \cdot \vec{p}_{e^+}) + (\vec{p}_{e^-} \cdot \vec{p}_{e^-}) + (\vec{p}_{e^+} \cdot \vec{p}_{e^-}) &= 0 \\ -m_e^2 - m_e^2 - m_e^2 &= 0 \implies m_e = 0! \end{aligned}$$

Since we know that m_e is in fact non-zero, this cannot possibly happen.

Now consider the scenario wherein two photons are created, moving in opposite directions. Then they would have momenta: $\vec{p}_{\gamma,1} \to_{\text{CM}} (h\nu, h\nu, 0, 0)$ and $\vec{p}_{\gamma,2} \to_{\text{CM}} (h\nu, -h\nu, 0, 0)$. Invoking conservation of four-momentum as before, we get

$$\begin{aligned} \vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,1} + \vec{p}_{\gamma,2} \\ (\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \cdot (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \\ -3m_e^2 &= (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,1}) + (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,2}) + (\vec{p}_{\gamma,2} \cdot \vec{p}_{\gamma,2}) \\ &= 0 + (-h^2\nu^2 - h^2\nu^2) + 0 = -2h^2\nu^2, \end{aligned}$$

so we end up with $3m_e^2 = 2h^2\nu^2$, meaning two photons are produced with $E^2 = \frac{3}{2}m_e^2$, which is entirely reasonable.

25

(a) Consider a frame $\bar{\mathcal{O}}$ moving with a speed v along the x-axis of $\mathcal{O}$. Now consider a photon moving at an angle θ from $\mathcal{O}$'s x-axis. Find the ratio of its frequency in $\bar{\mathcal{O}}$ and in $\mathcal{O}$.

We must first construct the particle's four-momentum. In the case where the photon was moving along the x-axis (see Section 2.7), it had been found that the four-momentum was

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E, 0, 0),$$

as this satisfied

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0. \qquad \text{(Schutz 2.37)}$$

Now that the photon is moving at an angle θ from the x-axis, we need to redistribute the 3-momentum accordingly. No specification was given as photon's angle in the y- or z-axis, so without loss of generality, I assume it is constrained to the x-y plane. This means we can write the four-momentum as

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E\cos\theta, E\sin\theta, 0),$$

which you can easily confirm satisfies $\vec{p}\cdot\vec{p}=0$.

Now we may apply the Lorentz transformation $\Lambda^{\bar{0}}{}_{\alpha}(v)$ to find the photon's energy as observed by $\bar{\mathcal{O}}$, and from that the frequency.

$$\begin{aligned} p^{\bar{0}} &= \bar{E} = \Lambda^{\bar{0}}{}_{\alpha}p^{\alpha} = \gamma p^0 - v\gamma p^1 + 0 + 0 = \gamma E - v\gamma E\cos\theta \\ \implies h\bar{\nu} &= \gamma h\nu - v\gamma h\nu\cos\theta \\ \implies \frac{\bar{\nu}}{\nu} &= \gamma - v\gamma\cos\theta = \frac{1-v\cos\theta}{\sqrt{1-v^2}} \end{aligned}$$

(b) Even when the photon moves perpendicular to the x-axis ($\theta=\pi/2$) there is a frequency shift. This is the *transverse Doppler shift*, which is a result of time dilation. At which angle θ must the photon move such that there is no Doppler shift between $\mathcal{O}$ and $\bar{\mathcal{O}}$?

To do this, we simply set $\bar{\nu}/\nu = 1$, and solve for θ.

$$\begin{aligned} 1 = \frac{1-v\cos\theta}{\sqrt{1-v^2}} &\implies \cos\theta = 1-\sqrt{1-v^2} \\ \implies \theta &= \pm\arccos\left(1-\sqrt{1-v^2}\right) \end{aligned}$$

(c) Now use Equations 2.35 and 2.38 to find $\bar{\nu}/\nu$.

Recall that $\vec{U} \rightarrow_{\mathcal{O}} (\gamma, v\gamma, 0, 0)$. Using Equation 2.35 we have

$$\begin{aligned} \bar{E} = h\bar{\nu} &= -(E, E\cos\theta, E\sin\theta, 0)\cdot(\gamma, v\gamma, 0, 0) \\ &= -(-(E\gamma) + E\gamma v\cos\theta) = E\gamma(1-v\cos\theta) = h\nu\gamma(1-v\cos\theta) \\ \frac{\bar{\nu}}{\nu} &= \frac{1-v\cos\theta}{\sqrt{1-v^2}} \end{aligned}$$

26 Calculate the energy required to accelerate a particle of rest mass $m>0$ from speed v to speed $v+\delta v$ ($\delta v \ll v$), to first order in δv. Show that it would take infinite energy to accelerate to c.

From the four-momentum we have $E_v = m\gamma$, and from that

$$E_{v+\delta v} = \frac{m}{\sqrt{1-(v+\delta v)^2}}.$$

If we do a Taylor expansion on $(1-(v+\delta v)^2)^{-1/2}$ we get

$$\frac{1}{\sqrt{1-v^2}} + \frac{v\,\delta v}{(1-v^2)^{3/2}} + \mathcal{O}\left(v^2\right),$$

so

$$\begin{aligned} E_{v+\delta v} &\approx \frac{m}{\sqrt{1-v^2}} + \frac{mv\,\delta v}{(1-v^2)^{3/2}} \\ \Delta E = E_{v+\delta v} - E_v &\approx \frac{mv\,\delta v}{(1-v^2)^{3/2}} = m\gamma^3 v\,\delta v\,. \end{aligned}$$

As $v\to c$, $\gamma\to\infty$ and therefore $\Delta E\to\infty$.

30 A rocket ship has four-velocity $\vec{U} \to_{\mathcal{O}} (2,1,1,1)$, and it passes a cosmic ray with four-momentum $\vec{p} \to \mathcal{O}(300, 299, 0, 0) \times 10^{-27}$kg. Compute the energy of the ray as measured by the rocket, using two different methods.

(a) Find the Lorentz transformation from $\mathcal{O}$ to the rocket's MCRF, and from that find the components $p^{\bar{\alpha}}$. The Lorentz transformation for a boost in the x, y, and z directions is given by

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

If we write out the terms of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

then we are left with a system of equations

$$\begin{aligned} 1 &= \gamma(2 + v_x + v_y + v_z), \\ 0 &= \gamma(2v_x + 1), \\ 0 &= \gamma(2v_y + 1), \\ 0 &= \gamma(2v_z + 1). \end{aligned}$$

Since γ may never be zero, we divide the last 3 terms by γ to obtain

$$2v_i + 1 = 0 \implies v_i = -\frac{1}{2},$$

and plugging into the first equation gives $\gamma = 2$. From this we see that our Lorentz transformation matrix is

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

Now to find the energy as observed by the rocket, we need to find $\bar{E} = p^{\bar{0}}$

$$\begin{aligned} p^{\bar{0}} &= \Lambda^{\bar{0}}{}_{\alpha} p^{\alpha} = 2p^0 - p^1 - p^2 - p^3 \\ &= (2 \cdot 300 - 1 \cdot 299 - 1 \cdot 0 - 1 \cdot 0) \times 10^{-27}\text{kg} = 3.01 \times 10^{-25}\,\text{kg} = \bar{E} \end{aligned}$$

(b) Use Schutz's Equation 2.35.

$$\begin{aligned}\bar{E} &= -\vec{p}\cdot\vec{U}_{\text{obs}} = -(-(300\cdot 2) + (299\cdot 1) + (0\cdot 1) + (0\cdot 1)) \times 10^{-27}\text{kg} \\ &= 3.01\times 10^{-25}\,\text{kg}\end{aligned}$$

(c) Which is quicker? Why?

Using Equation 2.35 was *much* quicker, as it was derived to handle this special case.

32 Consider a particle with charge e and mass m, which begins at rest, but scatters a photon with frequency ν_i (Compton scattering). The photon comes off at an angle θ from the direction of the initial photon's path. Use conservation of four-momentum to find the scattered photon's frequency, ν_f.

We will invoke: conservation of four-momentum and $\vec{p}\cdot\vec{p} = -m^2$. $\vec{p}_i$ and $\vec{p}_f$ denote the initial and final photon, and $\vec{p}_e$ and $\vec{p}_{e'}$ denote the electron before and after collision.

$$\begin{aligned}
\vec{p}_i &\underset{\mathcal{O}}{\to} (E_i, E_i, 0, 0) \\
\vec{p}_e &\underset{\mathcal{O}}{\to} (m, 0, 0, 0) \\
\vec{p}_f &\underset{\mathcal{O}}{\to} (E_f, E_f\cos\theta, E_f\sin\theta, 0) \\
\vec{p}_i + \vec{p}_e &= \vec{p}_f + \vec{p}_{e'} \\
\vec{p}_{e'} &= \vec{p}_i + \vec{p}_e - \vec{p}_f \\
\vec{p}_{e'}\cdot\vec{p}_{e'} &= (\vec{p}_i + \vec{p}_e - \vec{p}_f)\cdot(\vec{p}_i + \vec{p}_e - \vec{p}_f) \\
-m^2 &= \vec{p}_i\cdot\vec{p}_i + \vec{p}_e\cdot\vec{p}_e + \vec{p}_f\cdot\vec{p}_f + 2(\vec{p}_i\cdot\vec{p}_i - \vec{p}_i\cdot\vec{p}_f - \vec{p}_e\cdot\vec{p}_f) \\
&= 0 - m^2 + 0 + 2(\vec{p}_i\cdot\vec{p}_i - \vec{p}_i\cdot\vec{p}_f - \vec{p}_e\cdot\vec{p}_f) \\
0 &= \vec{p}_i\cdot\vec{p}_i - \vec{p}_i\cdot\vec{p}_f - \vec{p}_e\cdot\vec{p}_f \\
&= -E_i m - (-E_iE_f + E_iE_f\cos\theta) + E_f m \\
&= m(E_f - E_i) + E_iE_f(1-\cos\theta) \\
m(E_i - E_f) &= E_iE_f(1-\cos\theta) \\
mh(\nu_i - \nu_f) &= h^2\nu_i\nu_f(1-\cos\theta) \\
\frac{\nu_i - \nu_f}{\nu_i\nu_f} &= h\frac{1-\cos\theta}{m} \\
\frac{1}{\nu_f} - \frac{1}{\nu_i} &= h\frac{1-\cos\theta}{m} \\
\frac{1}{\nu_f} &= \frac{1}{\nu_i} + h\frac{1-\cos\theta}{m}
\end{aligned}$$