Spherical stars ⁴¹-21-21-2 ⁵⁰-21-2

Spherical solutions for stars

Daniel Wysocki

Rochester Institute of Technology

General Relativity I Presentatio December 14th, 2015

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Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- real stars

- I will model stars using GR assuming spherical symmetry
- I will derive the Schwarzschild metric and T–O–V equation
- finally I will look into specific types of stars

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Spherically symmetric coordinates

Spherically symmetric coordinates

• First we need to derive our coordinate system



2015-1

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Two-sphere in flat spacetime

General metric

 $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$

Spherical stars 2-14Spherically symmetric coordinates 2015-└─Two-sphere in flat spacetime

• we start with the simplest spherically symmetric coordinates

Two-sphere in flat spacetime

 $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

– flat spacetime

- 2-sphere in Minkowski spacetime
 - introduce $d\Omega^2$ for compactness

Schutz (2009, p. 256)

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Two-sphere in flat spacetime

General metric

 $\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$

Metric on 2-sphere

 $dl^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \equiv r^2 d\Omega^2$

Spherical stars 2-14Spherically symmetric coordinates 2015-1 └─Two-sphere in flat spacetime

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Two-sphere in flat spacetime

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- flat spacetime
- 2-sphere in Minkowski spacetime
 - introduce $d\Omega^2$ for compactness

Schutz (2009, p. 256)

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Two-sphere in curved spacetime

Metric on 2-sphere

 $\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$

$$f(r',t) \equiv r$$

Spherical stars 2-14Spherically symmetric coordinates 2015-1

└─Two-sphere in curved spacetime

• generalize to 2-sphere in arbitrary curved spherically symmetric spacetime

Two-sphere in curved spacetime

 $dl^2 = f(r', t)d\Omega$

• inclusion of curvature makes r^2 some function of r' and t

Schutz (2009, pp. 256–257)

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Two-sphere in curved spacetime

Metric on 2-sphere

 $\mathrm{d}l^2 = f(r', t)\mathrm{d}\Omega^2$

Relation to r

$$f(r',t) \equiv r^2$$

Spherical stars 2-14Spherically symmetric coordinates 2015-1 └─Two-sphere in curved spacetime

• generalize to 2-sphere in arbitrary curved spherically symmetric spacetime

Two-sphere in curved spacetime

Schutz (2009, pp. 256-257)

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Meaning of r

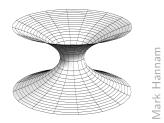


Figure:

• *not* proper distance from center

Spherical stars 14 Spherically symmetric coordinates Ċ) 2015-3 \square Meaning of r



- r is not necessary the "distance from the center"
- it is merely a coordinate "curvature" or "area" coordinate
- for instance, we may have a spacetime where the center is missing
 - example: Schwarzschild wormhole spacetime
- surface of constant (r, t) is a two-sphere of area A and circumference C

Surface with circular symmetry but no coordinate r = 0.

Schutz (2009, p. 257)

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Meaning of r

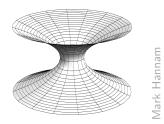
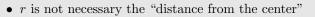


Figure:

• *not* proper distance from center

- "curvature" or "area" coordinate •
 - radius of curvature and area

Spherical stars 14 Spherically symmetric coordinates Ċ) 2015-3 \square Meaning of r



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Meaning of r

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Schutz (2009, p. 257)

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Meaning of r

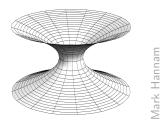


Figure:

Surface with circular symmetry but no coordinate r = 0.

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Spherical stars 14 Spherically symmetric coordinates Ċ) 2015-3 \square Meaning of r



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• r = const, t = const• $A = 4\pi r^2$ • $C = 2\pi r$

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Spherically symmetric spacetime

General metric

$$ds^{2} = g_{00} dt^{2} + 2g_{0r} dr dt + g_{rr} dr^{2} + r^{2} d\Omega^{2}$$

 $g_{00}, g_{0r}, \text{ and } g_{rr}$: functions of t and r

- now consider not only surface of 2-sphere, but whole spacetime
- now we have some unknown g_{00} , g_{rr} , and cross term g_{0r}
- cross term g_{0r}
- cross terms g_{0i} for $i \in \{\theta, \phi\}$ are zero from symmetry
- need more constraints to say anything particular about them

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$$\label{eq:gauge} \begin{split} \mathrm{d}s^2 &= g_{00}\,\mathrm{d}t^2 + 2g_{0v}\,\mathrm{d}r\,\mathrm{d}t + g_{trr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\\ g_{00},\,g_{0r},\,\mathrm{and}\,g_{trr};\,\mathrm{functions}\;\mathrm{of}\;t\;\mathrm{and}\;r \end{split}$$

Spherically symmetric spacetime

Spherical stars Static spacetimes

• now I will impose the static constraint

Static spacetimes

Static spacetimes



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Static spacetimes

Motivation

• leads to simple derivation of Schwarzschild metric

• unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem) Spherical stars -Static spacetimes -Motivation -Static spacetimes -Static spacetim

- we choose the constraint of a static spacetime because
 - it allows us to easily derive the Schwarzschild metric
 - according to Birkhoff's theorem, this metric is the unique solution to the Einstein vacuum field equations for spherically symmetric, asymptotically flat spacetimes
- George David Birkhoff

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843) Daniel Wysocki (RIT) Spherical stars December 14th, 2015 **AST** 9 / 41

Static spacetimes

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- leads to simple derivation of Schwarzschild metric
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Spherical stars - Static spacetimes - Motivation - Motivation - Motivation - Motivation

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Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843) Daniel Wysocki (RIT) Spherical stars December 14th, 2015

Sta	tic	spa	ceti	

Definition

A spacetime is static if we can find a time coordinate t for which

(i) the metric independent of t

 $g_{\alpha\beta,t} = 0$

(ii) the geometry unchanged by time reversal

 $t \rightarrow -i$

Definition Spherical stars 2-14Static spacetimes spacetime is static if we can find a time coordinate t for which i) the metric independent of t $g_{\alpha\beta,t} = 0$ 2015-1 -Definition

- now I define "static"
- first condition is that the metric is independent of time
 - by itself, this condition is called "stationary"
- second condition is that metric unaffected by time reversal
- e.g. rotating stars are stationary but not static

Schutz (2009, p. 258)

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Sta	tic.	spacetimes

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Definition Spherical stars 2-14Static spacetimes A spacetime is static if we can find a time coordinate t for which i) the metric independent of t $g_{\alpha\beta,t} = 0$ 2015-1 (ii) the geometry unchanged by time reversa -Definition

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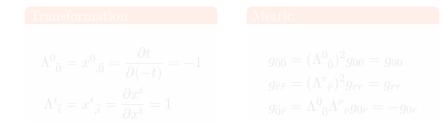
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Time reversal

$$\mathbf{\Lambda}:(t,x,y,z)\to (-t,x,y,z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}}\Lambda^{\beta}{}_{\bar{\beta}}g_{\alpha\beta} = g_{\alpha\beta}$$





- now I use the static constraint to simplify the metric
- transformation
 - -(0,0) term is dt/d(-t)
 - spatial terms are 1 if transformed to themselves
 - cross-terms are all zero, as coordinates independent of each other
- transformed metric
 - -(0,0) term is unchanged, as -1 is squared
 - (r, r) term is unchanged, as transformation is 1
 - (0, r) term is negated, but must still be equal, so it's zero
 - no cross terms

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Spherical stars

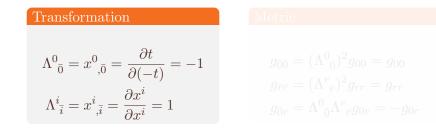
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Time reversal

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Time reversal

 $\Lambda^0_{\ \bar{0}} = x^0_{\ \bar{0}} = \frac{\partial t}{\partial (-t)} = -1$

 Λ : $(t, x, y, z) \rightarrow (-t, x, y, z)$

 $g_{\alpha\beta} = \Lambda^{\alpha}{}_{\dot{\alpha}}\Lambda^{\beta}{}_{\dot{\beta}}g_{\alpha\beta} = g_{\alpha\beta}$

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Static spacetimes

Time reversal

$$\mathbf{\Lambda}:(t,x,y,z)\to (-t,x,y,z)$$

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Transformation	Metric
$\Lambda^{0}{}_{\bar{0}} = x^{0}{}_{,\bar{0}} = \frac{\partial t}{\partial(-t)} = -1$ $\Lambda^{i}{}_{\bar{i}} = x^{i}{}_{,\bar{i}} = \frac{\partial x^{i}}{\partial x^{i}} = 1$	$g_{\bar{0}\bar{0}} = (\Lambda^{0}{}_{\bar{0}})^{2}g_{00} = g_{00}$ $g_{\bar{r}\bar{r}} = (\Lambda^{r}{}_{\bar{r}})^{2}g_{rr} = g_{rr}$ $g_{\bar{0}\bar{r}} = \Lambda^{0}{}_{\bar{0}}\Lambda^{r}{}_{\bar{r}}g_{0r} = -g_{0r}$



- now I use the static constraint to simplify the metric
- transformation
 - (0,0) term is dt/d(-t)
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Time reversal

 $\Lambda^0_{\bar{n}} = x^0_{\bar{n}} = \frac{\partial t}{\partial t}$

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 $g_{\alpha\beta} = \Lambda^{\alpha}{}_{\dot{\alpha}}\Lambda^{\beta}{}_{\dot{\beta}}g_{\alpha\beta} = g_{\alpha\beta}$

 $g_{30} = (\Lambda^0_{\ b})^2 g_{30} = g_{30}$

- transformed metric
 - -(0,0) term is unchanged, as -1 is squared
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 - $\bullet~$ no cross terms

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Static spacetimes

Simplified metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$a_{00} \rightarrow -e^{2\Phi}, \quad a_{rr} \rightarrow e^{2\Lambda}, \quad \text{provided } a_{00} < 0 < 0$$

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Spherical stars 2-14Static spacetimes 2015-└─The metric



- now we simplify the metric, since the cross term is zero
- we assume g_{00} to be negative, and g_{rr} to be positive
 - signature is (-, +, +, +)
 - holds inside stars but not black holes
- limits at infinity tell us that spacetime is asymptotically flat $-\Phi = \Lambda = 0 \implies e^{2\Phi} = e^{2\Lambda} = 1 \text{ and } \mathbf{g} = \eta$

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Static spacetimes	
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Simplified metric

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + g_{rr}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Replacement

$$g_{00} \rightarrow -e^{2\Phi}, \quad g_{rr} \rightarrow e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Spherical stars 2-14Static spacetimes 2015-1 └─The metric



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Schutz (2009, pp. 258–259)

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Static spacetimes	
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Static spherically symmetric metric

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\lim_{r \to \infty} \Phi(r) = \lim_{r \to \infty} \Lambda(r) = 0$$

Schutz (2009, pp. 258–259)

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 $\begin{array}{c} {\rm Spherical stars} \\ {\rm Fr}_{\rm CI} \\ {\rm Spherical stars} \\$

The metric Complete variant $d^2 = g_{00}d^2 + g_{00}d^2 + g^2 + d^2 d^2$ Formation $g_{00} = e^{i\theta}$, $g_{00} + e^{i\theta}d^2$, formaling $g_{00} < 0 < g_{00}$ Complete and $g_{00} = e^{i\theta}d^2 + d^2 + d^2 + d^2 + d^2$ $d_{00} = e^{i\theta}d^2 + d^2 + d^2 + d^2 + d^2$ $d_{00} = e^{i\theta}d^2 + d^2 + d^2 + d^2 + d^2$ $d_{00} = e^{i\theta}d^2 + d^2 + d^2 + d^2 + d^2$

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Static spacetimes

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Einstein Tensor

General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda} (1 - e^{-2\Lambda}) + \frac{2}{r} \Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}$$

Spherical stars 2-14Static spacetimes 2015--Einstein Tensor

- now we can use the metric to derive the Riemann tensor
- from that the Einstein tensor
- the derivation is involved, so we will just take them as is
- we're going to use some of these components later on
- note that *prime* denotes d/dr

Schutz (2009, pp. 165, 260)

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Einstein Tensor

Schutz (2009, pp. 165, 260)

 $G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$

Static spacetimes

Einstein Tensor

General Einstein tensor

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Einstein tensor components

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})]$$

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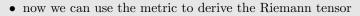
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Einstein Tensor

 $G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$

 $G_{00} = \frac{1}{z^2} e^{2\Phi} \frac{d}{dz} [r(1 - e^{-2\Lambda})]$

Schutz (2009, pp. 165, 260)

 $G_{rr} = -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi'$ $G_{\theta\theta} = r^2e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$

- from that the Einstein tensor
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Spherical stars -Static perfect fluid

Static perfect fluid

- stars are fluids for simplicity we assume perfect
- thus we will impose additional constraints accordingly



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Static perfect fluid	S
Four-velocity	2015-12-14
	2015
Constraints	
$U^i = 0$ (static) $\vec{U} \cdot \vec{U} = -1$ (conservation law)	

 Spherical stars
 Four-velocity

 Four-velocity
 Image: Spherical stars

- static fluid, so in MCRF three-velocity components all zero
- we find the only non-zero term, U^0 , by relating to the dot product
- lower it with the metric, to use in next part

$$g_{00}U^{0}U^{0} = -1 \implies (U^{0})^{2} = (-g_{00})^{-1}$$
$$\implies U^{0} = (-g_{00})^{-1/2}$$
$$\implies U^{0} = (e^{2\Phi})^{-1/2} = e^{-\Phi}$$

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Schutz (2009, p. 260)

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Static perfect fluid				
Four-velocity				
Constraints				
Constraints				
$U^i = 0$ (static) $\vec{U} \cdot \vec{U} = -1$ (conservation law)				
Solving for U^0				
$g_{00}U^0U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$				

Solving for U_0

$$U_0 = g_{00} U^0 = -e^{\frac{1}{2}}$$

Schutz (2009, p. 260)

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- static fluid, so in MCRF three-velocity components all zero
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Four-velocity

Schutz (2009, p. 290)

 $\vec{U} \cdot \vec{U} = -1$ (conservation law)

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Static perfect fluid			
Four-velocity			
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Solving for U_0

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Schutz (2009, p. 260)

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Stress–energy tensor

Stress-energy tensor for perfect fluid

$$T_{\alpha\beta} = (\rho + p)U_{\alpha}U_{\beta} + pg_{\alpha\beta}$$

Components of $T_{lphaeta}$





- $T_{i\alpha} = pg_{i\alpha}$ because spatial components of U are zero
- $T_{\alpha\beta}$ is diagonal because of previous condition and $g_{\alpha\beta}$ is diagonal
- T_{00} requires a little algebra
- T_{ii} just need to multiply metric by p
- $T_{\phi\phi}$ can be written in terms of $T_{\theta\theta}$

Schutz (2009, p. 260)

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T_{00}	T_{0r}	$T_{0\theta}$	$T_{0\phi}$
T_{r0}	T_{rr}	$T_{r\theta}$	$T_{r\phi}$
$T_{\theta 0}$	$T_{\theta r}$	$T_{\theta\theta}$	$T_{\theta\phi}$
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Spherical stars 2-14Static perfect fluid 2015--Stress-energy tensor



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Schutz (2009, p. 260)

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 $\begin{bmatrix} T_{00} & T_{0r} & T_{0\theta} & T_{0\phi} \\ T_{r0} & T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta0} & T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi0} & T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{bmatrix}$



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Static perfect fluid

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- $\begin{aligned} & \text{Rimes-energy tensor} \\ \hline \\ & \text{Remember 2. Hence for each Read.} \\ & \mathcal{T}_{ab} = (e + p) \mathcal{U}_{ab} \mathcal{U}_{ab} + p_{abc}. \\ \hline \\ & \text{Remember 2. Hence for each Read.} \\ & \text{Remember 2. Hence for each Read.} \\ & \text{T}_{ab} = p_{abc} + p_{ab}^{abb} = e_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc} + p_{abc}^{abb} = e_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} = p_{abc}^{abb} \\ & \text{T}_{abc} = p_{abc}^{abb} + p_{abc}^{abb} \\ & \text{T}_{abc}^{abb} + p_{abc}^{abb} \\ & \text{T}_{abc}^{abb} = p_{abc}^{abb} + p_{abc}^{abb} \\ & \text{T}_{abc}^{abb} + p_{abc}^{abb} \\ & \text{T}_{abc}^{abb} + p_{abc}^{abb} \\ & \text{T}_{abc}^{abb} + p_{abc}^{abb}$
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Equation of state

Local thermodynamic equilibrium

 $p = p(\rho, S) \approx p(\rho)$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

- Spherical stars 2-14Static perfect fluid 2015-1 -Equation of state
 - in a static fluid we have local thermodynamic equilibrium

Equation of state

· pressure related to energy density and specific entrop

we often deal with negligibly small entropies

- pressure a function of density and specific entropy
- specific entropy assumed negligibly small

Schutz (2009, p. 261)

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Static perfect fluid

Equations of motion

Conservation of 4-momentum

 $T^{\alpha\beta}_{\ \ ;\beta} = 0$

• symmetries make only non-trivial solution $\alpha = r$

Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}$$

Spherical stars ⁴-Static perfect fluid ⁴-Static perfect fluid ⁴-Equations of motion

- first equation follows from conservation of 4-momentum
- due to symmetry, the only non-trivial solution is for $\alpha = r$

Equations of motion

- equation of motion for static perfect fluid
- (derivation in bonus slides)

Schutz (2009, pp. 175, 261)

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Static perfect fluid

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Spherical stars 2-14Static perfect fluid 2015-1 -Equations of motion

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Equations of motion

Schutz (2009, pp. 175, 261)

symmetries make only non-trivial solution α = r

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St	atic	perfect	fluid

Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})] = 8\pi\rho e^{2\Phi}$$

$$n(r) \equiv \frac{1}{2}r(1-e^{-2\Lambda})$$
 or $g_{rr} = e^{2\Lambda} \equiv \left(1-\frac{2m(r)}{r}\right)^{-1}$

 Relation to energy density

 $\frac{dm(r)}{dr} = 4\pi r^2 \rho$

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- inspect (0,0) component of Einstein equations
- define the mass function, m(r)
- in Newtonian limit, m(r) is mass within radius r

 $m(r) = 4\pi \int_0^r (r')^2 \rho(r') \,\mathrm{d}r'$



Static perfect fluid				
	- St	atic	perfect	fluid

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Schutz (2009, pp. 260–262)

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2$$

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Static	perfect	

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Spherical stars Static perfect fluid Static Defect fluid Mass function

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Static		
	perfect	

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$$m(r)$$
 .

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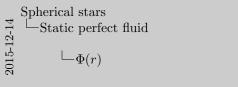
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- Spherical stars -Static perfect fluid -Static perfect fluid -Mass function
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Static perfect fluid
$\Phi(r)$
Einstein field equations
$G_{rr} = 8\pi T_{rr} \implies -\frac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi p e^{2\Lambda}$
$\Phi(r)$
$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$



- inspect (r, r) component of Einstein equations
- gives us an expression for $\Phi(r)$

Schutz (2009, pp. 260–262)

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 $G_{zz} = 8\pi T_{zz} =$

Schutz (2009, pp. 260-262)

 $\Phi(r)$

	atic perfect fluid	
$\Phi(m)$		

Spherical stars 12 - 14-Static perfect fluid 2015-1 $\Box \Phi(r)$

• inspect (r, r) component of Einstein equations

 $\Phi(r)$

Schutz (2009, pp. 260-262)

 $G_{rr} = 8\pi T_{rr} \implies -\frac{1}{z^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{z}\Phi' = 8\pi pe^{2\Lambda}$

• gives us an expression for $\Phi(r)$

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Schutz (2009, pp. 260–262)

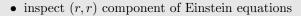
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Spherical stars 12 - 14-Static perfect fluid 2015-1 $\Box \Phi(r)$



 $\Phi(r)$

Schutz (2009, pp. 260-262)

 $G_{rr} = 8\pi T_{rr} \implies -\frac{1}{a^2}e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{a}\Phi' = 8\pi pe^{2\Lambda}$

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Schutz (2009, pp. 260–262)

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Spherical stars —Exterior Geometry

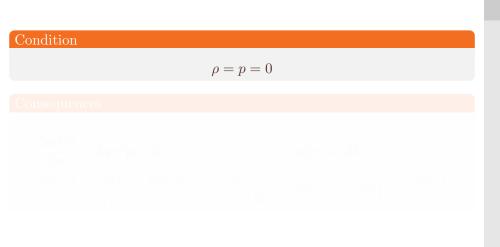
Exterior Geometry

- until now, we've not considered whether we were inside or outside star
- properties inside different than outside (obviously)
- we're going to inspect both cases, starting with outside



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Schwarzschild metric I



Spherical stars Exterior Geometry Schwarzschild metric I

- the external conditions just state we are in a vacuum
 - $\,-\,$ breaks down when matter surrounds star
- m(r) is constant, we call it M
- $d\Phi/dr$ simplifies, and we can now integrate it to find $\Phi(r)$

Schwarzschild metric I

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Schwarzschild metric I

 $\rho = p = 0$

Consequences

Condition

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad \qquad m(r) \equiv M$$
$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r-2m(r)]} = \frac{M}{r(r-2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

Spherical stars 2-14Exterior Geometry 2015--Schwarzschild metric I

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Spherical stars Exterior Geometry

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Schutz (2009, pp. 262–263)

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Schwarzschild metric I

 $\frac{dm(r)}{dr} = 4\pi r^2 \rho = 0$

 $\frac{\frac{d\theta(r)}{dr}}{\frac{d\theta(r)}{dr}} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)}$ Schutz (2009, pp. 262–263)

 $m(r) \equiv M$

Schutz (2009, pp. 262–263)

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Schwarzschild metric I

Condition

 $\rho = p = 0$

Consequences

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad \qquad m(r) \equiv M$$
$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

Spherical stars Exterior Geometry Schwarzschild metric I

- the external conditions just state we are in a vacuum
 - breaks down when matter surrounds star
- m(r) is constant, we call it M
- $d\Phi/dr$ simplifies, and we can now integrate it to find $\Phi(r)$

Schwarzschild metric I

 $\frac{dm(r)}{dr} = 4\pi r^2 \rho = 0$

 $\frac{d\Phi(r)}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} =$

Schutz (2009, pp. 262-263)

 $\frac{M}{r(r-2M)} = \Phi(r) = \frac{1}{2} \log \left(1 - \frac{2M}{r}\right)$

Schutz (2009, pp. 262–263)

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Schwarzschild metric II

First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$$
 $g_{00} = -e^{2\Phi} = -\left(1 - \frac{2}{r}\right)^{-1}$

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\Omega^{2}$$

Spherical stars 2-14Exterior Geometry 2015--Schwarzschild metric II

- recall g_{rr} from earlier
- substituting our expression from $\Phi(r)$ into $-e^{2\Phi}$ gives g_{00}

Schwarzschild metric II

 $g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$

Schutz (2009, pp. 258, 262-263)

• we have found the Schwarzschild metric!

Schutz (2009, pp. 258, 262–263)

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Schwarzschild metric II

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Spherical stars 2-14Exterior Geometry 2015--Schwarzschild metric II

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Schwarzschild metric II $g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$ $g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$ Schutz (2009, pp. 258, 262-263)

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Schwarzschild metric

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$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\Omega^{2}$$

Spherical stars 2-14Exterior Geometry 2015--Schwarzschild metric II

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Schutz (2009, pp. 258, 262-263)

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Schutz (2009, pp. 258, 262–263)

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	Exteri	ior C	leomet	rv
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Condition

 $r \gg M$

Schwarzschild metric

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \left(1 - \frac{2M}{r}\right)^{-1}dr^{2} + r^{2}d\Omega^{2}$$

Far-field Schwarzschild metric (Cartesian)

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$R^2 \equiv x^2 + y^2 + z^2$

Schutz (2009, pp. 263)

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Spherical stars

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Spherical stars Exterior Geometry Far-field metric

- far-field metric of a star (far away)
- consider m(r) to be total mass M
- can use Taylor expansion, and to first order rewrite as such
- $\bullet\,$ we can define a new coordinate R, the distance from the star

Far-field metric

- Cartesian coordinates

Exterior Geometry

Condition

 $r \gg M$

Far-field Schwarzschild metric

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Schutz (2009, pp. 263)

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Spherical stars Exterior Geometry Far-field metric

- Far field metric $\label{eq:result} \begin{aligned} F &= far field metric \\ \hline \\ \hline \\ &= \frac{c \cos(2\pi i t)}{2} \frac{c^2}{2} + \frac{c^2}{2} \frac{c^2}{2$
- far-field metric of a star (far away)
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Exterior Geometry	
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Condition

 $r \gg M$

Far-field Schwarzschild metric

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{r}\right) - \mathrm{d}r^2 + r^2\,\mathrm{d}s^2$$

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

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Spherical stars 2-14Exterior Geometry 2015-

Far-field metric

- far-field metric of a star (far away)
- consider m(r) to be total mass M
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Far-field metric

 $r \gg M$

 $ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) - dr^2 + r^2 d\Omega^2$

- we can define a new coordinate R, the distance from the star
 - Cartesian coordinates

	Geometry
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Condition

 $r \gg M$

Far-field Schwarzschild metric

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{r}\right) - \mathrm{d}r^2 + r^2\,\mathrm{d}s^2$$

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Schutz (2009, pp. 263)

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Spherical stars

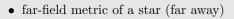
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Spherical stars 2-14Exterior Geometry 2015-

└─Far-field metric



- consider m(r) to be total mass M
- can use Taylor expansion, and to first order rewrite as such
- we can define a new coordinate R, the distance from the star
 - Cartesian coordinates



- 2015-12-14
- Spherical stars —Interior structure

Interior structure

- now we look at the remaining, and most interesting regime
 - $-\,$ inside the star
- our assumptions from outside the star no longer hold



Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

 $\rho \neq 0 \quad p \neq 0$

Spherical stars Interior structure

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└─Tolman–Oppenheimer–Volkov (T–O–V) equation

Tolman-Oppenheimer-Volkov (T-O-V) equation

$\rho \neq 0 p \neq 0$

- inside a star, we cannot assume density and pressure are zero
- revisit two earlier equations
- substitute one into the other
- arrive at the T–O–V equation
- gives us an ODE relating
 - pressure p
 - density ρ
 - mass function m(r)
 - radius r
 - eventually hope to solve all quantities in terms of r

Schutz (2009, pp. 261–264)

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Spherical stars

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Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

$\rho \neq 0 \quad p \neq 0$ Recall $(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \qquad \qquad \frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$ T-O-V contains

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r)+4\pi r^3 p]}{r[r-2m(r)]}$$

Schutz (2009, pp. 261–264)

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 $\begin{array}{c} \text{Spherical stars} \\ \stackrel{\bullet}{\overset{\bullet}{\underset{\bullet}}} & \stackrel{\bullet}{\overset{\bullet}{\overset{\bullet}}} \text{Interior structure} \end{array}$

2015-

└─Tolman–Oppenheimer–Volkov (T–O–V) equation Tolman–Oppenheimer–Volkov (T–O–V) equation

	$\rho \neq 0$	$p \neq 0$	
call			
$(\rho + p)\frac{d\Phi}{dr} = -$	$\frac{dp}{dr}$		

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Tolman–Oppenheimer–Volkov (T–O–V) equation

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$\rho \neq 0 \quad p \neq 0$ Recall $(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \qquad \qquad \frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$ T-O-V equation

$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r)+4\pi r^3 p]}{r[r-2m(r)]}$

Schutz (2009, pp. 261–264)

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Spherical stars

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 $\begin{array}{c} \text{Spherical stars} \\ \stackrel{\bullet}{\overset{\bullet}{\underset{\bullet}}} & \stackrel{\bullet}{\overset{\bullet}{\underset{\bullet}}} \text{Interior structure} \end{array}$

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└─Tolman–Oppenheimer–Volkov (T–O–V) equation Tolman–Oppenheimer–Volkov (T–O–V) equation

	$\rho \neq 0 p \neq 0$
Recall	
$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}$	$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$

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Tolman–Oppenheimer–Volkov (T–O–V) equation

Condition

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r)+4\pi r^3 p]}{r[r-2m(r)]}$$

Schutz (2009, pp. 261–264)

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 $\begin{array}{c} \text{Spherical stars} \\ \stackrel{\bullet}{\overset{\bullet}{\underset{\bullet}}} & \stackrel{\bullet}{\overset{\bullet}{\overset{\bullet}}} \text{Interior structure} \end{array}$

2015 -

└─Tolman–Oppenheimer–Volkov (T–O–V) equation Tolman–Oppenheimer–Volkov (T–O–V) equation

	$\rho \neq 0$	$p \neq 0$	
Recall			
$(\rho + p)\frac{d\Phi}{dr} = -$	dp dr	$\frac{d\Phi}{dr}$,	$= \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$
-O-V equation			
$\frac{d\mu}{dr}$	$= -\frac{(\rho + p)}{r}$	$[m(r) + 4\pi r^3]$ [r - 2m(r)]	<u>el</u>

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System of coupled differential equations

T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)[m(r)+4\pi r^3 p]}{r[r-2m(r)]}$$

Mass function

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2$$

 $\mathbf{\Omega}$

Equation of state

$p = p(\rho)$

Schutz (2009, pp. 261–262, 264)

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 $\begin{array}{c} \text{Spherical stars} \\ \stackrel{\bullet}{\overset{\bullet}{\underset{\bullet}}} & \stackrel{\bullet}{\overset{\bullet}{\overset{\bullet}}} \text{Interior structure} \end{array}$

2015-

System of coupled differential equations

System of coupled differential equations

T-O-V equation	
	$\frac{dp}{dr} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$
Mass function	
	$\frac{dm(r)}{dr} = 4\pi r^2 \rho$
Equation of state	
	$p = p(\rho)$

- T–O–V equation coupled with $\mathrm{d}m/\mathrm{d}r$ and $p(\rho)$
 - 3 equations
 - 3 unknowns (m, ρ, p)
 - $\Phi(r)$ only intermediate variable
- can integrate to find m(r), $\rho(r)$, and p(r)

Newtonian hydrostatic equilibrium

Newtonian limit

 $p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -$$

Spherical stars $\stackrel{\text{T}}{\longrightarrow}$ Interior structure

2015 -

└─Newtonian hydrostatic equilibrium

- in the Newtonian limit we get these constraints
- which allow us to cancel terms in the T–O–V equation

Newtonian hydrostatic equilibrium

 $p \ll \rho$: $4\pi r^3 p \ll m$; $m \ll r$

Schutz (2009, pp. 265-206) and Bansen and Kawaler (1994, p. 3

• and arrive at the familiar equation of HSE

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)

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Newtonian hydrostatic equilibrium

Newtonian limit

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll$$

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -$$

Spherical stars 2-14Interior structure

2015-

—Newtonian hydrostatic equilibrium

• in the Newtonian limit we get these constraints

Newtonian hydrostatic equilibrium

 $p \ll \rho$: $4\pi r^3 p \ll m$; $m \ll r$

 $\frac{\rho}{r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$

Schutz (2009, pp. 265-296) and Bansen and Kawaler (1994, p. 3

- which allow us to cancel terms in the T–O–V equation
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Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3) Spherical stars

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Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

Spherical stars $\stackrel{\text{Pi}}{\leftarrow}$ Interior structure

2015-

└─Newtonian hydrostatic equilibrium

Newtonian hydrostatic equilibrium $\label{eq:hydrostatic equilibrium} \\ \begin{array}{l} \textbf{Newtonian} \\ p < \mu - 4\pi^2 p < m & m < r \end{array} \\ \\ \begin{array}{l} \textbf{P} = \mu - 4\pi^2 p < m & m < r \end{array} \\ \\ \begin{array}{l} \textbf{P} = \mu - \frac{(\mu + \mu)[m(t) + 4\pi^2 \mu]}{r^2 - 2m(t)]} = -\frac{m(t)}{r^2} \end{array} \\ \end{array}$

Schutz (2009, pp. 265–206) and Hansen and Kawaler (1994, p. 3

- in the Newtonian limit we get these constraints
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- and arrive at the familiar equation of HSE

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3) Daniel Wysocki (RIT) Spherical stars December 14th, 2015

Constant density solution I

Constraint

 $\rho(r) \equiv \rho_0$

$$m(r) = \frac{4}{3}\pi\rho_0 \begin{cases} r^3, & r \le R\\ R^3, & r \ge R \end{cases}$$

Schutz (2009, pp. 266-267)

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Spherical stars 14 Interior structure à 2015-

-Constant density solution I

Constant density solution I

- because it is the simplest case, we are going to investigate a star of uniform density, $\rho(r) \equiv \rho_0$
 - this is unphysical
 - for instance, the speed of sound in such a star is infinite
 - neutron star density is *almost* uniform
 - also leads us to a result which holds for all stellar densities
- easy to obtain mass function from earlier differential equation
 - equal to the density times the volume of the sphere enclosed by radius r inside
 - equal to the density times the volume of the entire star (r = R)when outside
 - continuous at the boundary

Constant density solution I

Constraint

 $\rho(r) \equiv \rho_0$

Mass function

$$m(r) = \frac{4}{3}\pi\rho_0 \begin{cases} r^3, & r \le R\\ R^3, & r \ge R \end{cases}$$

Schutz (2009, pp. 266-267)

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- Spherical stars 14 Interior structure à 2015-
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Constant density solution I

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 - continuous at the boundary

Constant density solution II

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)} = -\frac{4}{3}\pi r \frac{(\rho_0+p)(\rho_0+3p)}{1-\frac{8}{3}r^2\rho_0}$$

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c} \sqrt{1 - 2m/r}$$

Spherical stars 14 Interior structure Ċ) 2015--Constant density solution II

• recall the T–O–V equation, which describes the interior of the star

Constant density solution II

 $\frac{\mathrm{d}p}{\mathrm{d}r}=-\frac{(\rho+p)(m+4\pi r^3p)}{r(r-2m)}$

- we can substitute m(r) for r < R, to simplify it as shown
- this gives us a separable differential equation
- we integrate the differential equation from the center $(r = 0, p = p_c)$ to some radius (r = r, p = p)
- to simplify the expression again, we've re-written it in terms of m(r)
- now we have a relation between ρ_0 , p, and m(r) at a given r

Schutz (2009, pp. 264, 266-267)

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Constant density solution II

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)} = -\frac{4}{3}\pi r \frac{(\rho_0+p)(\rho_0+3p)}{1-\frac{8}{3}r^2\rho_0}$$

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Spherical stars 14 Interior structure Ċ) 2015--Constant density solution II

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Schutz (2009, pp. 264, 266-267)

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 $\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} = -\frac{4}{3}\pi r \frac{(\rho_0 + p)(\rho_0 + 3p)}{1 - \frac{8}{9}r^2\rho_0}$

Constant density solution II

Constant density solution II

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho+p)(m+4\pi r^3 p)}{r(r-2m)} = -\frac{4}{3}\pi r \frac{(\rho_0+p)(\rho_0+3p)}{1-\frac{8}{3}r^2\rho_0}$$

Integrated from center to internal radius r

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c} \sqrt{1 - 2m/r}$$

Spherical stars 14 Interior structure Ċ) 2015--Constant density solution II

• recall the T–O–V equation, which describes the interior of the star

Constant density solution II

Schutz (2009, pp. 264, 266-267)

 $\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} = -\frac{4}{3}\pi r \frac{(\rho_0 + p)(\rho_0 + 3p)}{1 - \frac{8}{9}r^2\rho_0}$

 $\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c} \sqrt{1 - 2m/r}$

- we can substitute m(r) for r < R, to simplify it as shown
- this gives us a separable differential equation
- we integrate the differential equation from the center $(r = 0, p = p_c)$ to some radius (r = r, p = p)
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Interior structure

Constant density solution III

Radius R

$$R^{2} = \frac{3}{8\pi\rho_{0}} \left[1 - \left(\frac{\rho_{0} + p_{c}}{\rho_{0} + 3p_{c}}\right)^{2} \right]$$

Central pressure p

$$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$$

Limit on M/R

$M/R \to 4/9 \implies p_c \to \infty$

Schutz (2009, pp. 266-267, 269)

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Spherical stars

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Spherical stars $\stackrel{\text{T}}{\to}$ —Interior structure

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Constant density solution III

- at the surface, r = R and p = 0
- $\bullet\,$ can solve the previous equation for R
- from this, we can solve for p_c
 - this gives us an expression for the central pressure necessary
- we can see that this blows up when M/R = 4/9

 $3\sqrt{1-8/9} - 1 = 3\sqrt{1/9} - 1 = 1 - 0 = 0$

- radius cannot be smaller than (9/4)M
 - less than the 2M needed for a black hole
- Buchdahl's theorem states that this is true in general for all stars
 - not just $\rho(r) \equiv \rho_0$

Constant density solution III $R^{2} = \frac{1}{8\pi \rho_{0}} \left[1 - \left(\frac{\rho_{0} + \rho_{0}}{\rho_{0} + 2\rho_{0}} \right)^{2} \right]$ $R^{2} = \frac{1}{8\pi \rho_{0}} \left[\frac{1}{2} - \left(\frac{\rho_{0} + \rho_{0}}{2} \right)^{2} - \frac{1}{2} \right]$ $R^{2} = \frac{1}{2} - \frac{1}{2$

Interior structure

Constant density solution III

Radius R

$$R^{2} = \frac{3}{8\pi\rho_{0}} \left[1 - \left(\frac{\rho_{0} + p_{c}}{\rho_{0} + 3p_{c}}\right)^{2} \right]$$

Central pressure p_c

$$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$$

Limit on M/R

$M/R \to 4/9 \implies p_c \to \infty$

Schutz (2009, pp. 266-267, 269)

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• even for non-constant density, M/R < 4/9

Spherical stars 2-14Interior structure 2015-3 -Buchdahl's theorem

- restate M/R < 4/9 from Buchdahl's theorem
- give Carroll's intuitive explanation
 - if we assume there is a maximum sustainable density in nature
 - and we consider an object which fills a sphere with radius R
 - then the most massive possible object within that volume would have a uniform density
 - all other objects would need to have a lower density

Carroll (2004, pp. 234)

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Buchdahl's theorem

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Spherical stars 14 Interior structure Ċ) 2015--Buchdahl's theorem

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Spherical stars 14 Interior structure Ċ) 2015--Buchdahl's theorem

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- even for non-constant density, M/R < 4/9
- intuitive explanation:
 - assume there is a maximum sustainable density, $(M/R)_{\text{max}}$
 - consider an object of radius R
 - most massive possible object would have maximum density everywhere

Spherical stars 14 Interior structure Ċ) 2015-3 -Buchdahl's theorem

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 - all other sustainable objects have a lower M/R

Spherical stars 14 Interior structure 2 2015-3 -Buchdahl's theorem

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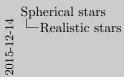
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Realistic stars

• now we're going to have a brief overview of real stars

Realistic stars



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ealistic stars

White dwarfs

- end-of-life for low mass stars
- held up by electron degeneracy pressure
- Newtonian structure accurate to 1%

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m}{r^2}$$

• relativistic effects important on stability and pulsation for

$$10^8 {\rm g} {\rm \, cm}^{-3} \lesssim \rho_c \lesssim 10^{8.4} {\rm g} {\rm \, cm}^{-3}$$

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 White dwarfs • end-of life for low mass stars • hold up by electron degeneracy pressure • Norwtonian structure accented to 1%, $\frac{dp}{dr} = -\frac{dp}{r^2}$ • relativeite effects important on a stability and pathston $H^0 g cm^3 \lesssim \rho \lesssim 10^4 k g cm^3$ - Money, Theren, and Warker (UT), $\rho(T)$

- end-of-life form of lower mass stars like our Sun is as a white dwarf
- core left over after a star loses its outer shell as a planetary nebula
- nuclear fusion has halted, and only pressure of degenerate electron gas supports them
 - Pauli exclusion principle
- structure can be described by the equation of HSE to high accuracy
- relativistic effects come into play for central densities:
 - $\ {\rm over} \ 10^8 {\rm g} \, {\rm cm}^{-3}$
 - up until the maximum

Neutron stars

- mass condensed further than white dwarf
- created in supernovae, or collapse of white dwarf

 $p^+ + e^- \rightarrow n^0 + \nu$

- held up by neutron degeneracy pressure
- matter incredibly complex and possess many unknown properties

- Spherical stars 14 Realistic stars 2 2015-1└─Neutron stars
 - when a star condenses beyond a white dwarf, it may become a neutron star

Neutron stars

mass condensed further than white dwarf

 held up by neutron degeneracy pressure matter incredibly complex and possess many unknown propertie

created in supernovae, or collapse of white dwar

- occurs in the aftermath of a supernova, or collapse of white dwarf
- compression beyond neutron star would form a black hole
- kinetic energy of electrons high
 - allows energy release when combined with a proton
 - energy carried away by neutrino, and neutron left behind
- held up by neutron degeneracy pressure Pauli again
- matter incredibly complex
 - suitable equation of state is a topic under active research

Schutz (2009, pp. 274–275)

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Rotating stars

Metric

$ds^{2} = -e^{2\nu} dt + e^{2\psi} (d\phi - \omega dt)^{2} + e^{2\mu} (dr^{2} + r^{2} d\theta^{2}),$

ν, ψ, ω , and μ : functions of r and θ

stationary •

• can still assume perfect fluid to high accuracy

Spherical stars 2-14Realistic stars 2015--Rotating stars

• much more complicated when we allow for rotation

Rotating stars

 $ds^2 = -e^{2\nu} dt + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\mu} (dr^2 + r^2 d\theta^2)$

can still assume perfect fluid to high accuracy

- metric no longer static
 - addition of cross terms between t and ϕ
 - metric dependence on θ in addition to r
- metric is still stationary
- perfect fluid assumption works to high accuracy

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Pulsars

- rapidly rotating neutron stars
- magnetic field produces electromagnetic radiation
- pulses of radio waves observed with the right orientation
- introduction of strong magnetic field requires
 - consideration of coupled Einstein–Maxwell field equations
 - $T_{\alpha\beta}$ includes EM energy density non-isotropic

Misner, Thorne, and Wheeler (1973, p. 628) and Stergioulas (2003, p. 28) December 14th, 2015

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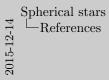
Spherical stars Realistic stars Ċ) 2015-L–Pulsars

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- pulsars are rapidly rotating neutron stars
- they have a strong magnetic field which causes emission of light
- magnetic poles may be offset from axis of rotation
- if observed from right angle, see pulses of radio light, like lighthouse
- by including a strong magnetic field, we need to
 - consider the coupled Einstein–Maxwell field equations, assuming
 - equilibrium
 - stationary
 - axisymmetric
 - internal electric current
 - need to include electromagnetic energy density to stress-energy tensor
 - this makes $T_{\alpha\beta}$ non-isotropic



• You made it to the end!

References



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References

ferences

 $\begin{array}{c} \text{Spherical stars} \\ \stackrel{\text{T}}{\leftarrow} \text{References} \\ \stackrel{\text{T}}{\leftarrow} \end{array}$

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Spherical stars Bonus slides

Bonus slides

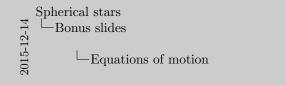
Bonus slides



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Equations of motion



Equations of motion

 $T^{\alpha\beta}_{\ \ \beta} = 0$, $T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + pg^{\alpha\beta}$ $T^{r\beta}_{\ \beta} = (\rho + p)U^{\beta}U^{r}_{\ \beta} + g^{rr}p_{,r} = 0$ $= (\rho + p)U^{\beta}U^{\lambda}\Gamma^{r}_{\lambda\beta} + e^{-2\Lambda}p_{,r} = 0$ $= (\rho + p)(U^0)^2 \Gamma'_{00} + e^{-2\Lambda} p_F = 0$ $= (\rho + p)(e^{-2\Phi})(e^{-2\Lambda}e^{2\Phi}\Phi_{,r}) + e^{-2\Lambda}p_{,r} = 0$ $-\frac{dp}{dr} = (\rho + p)\frac{d\Phi}{dr}$

Schutz (2009, pp. 101, 261)

$$\begin{split} T^{\alpha\beta}_{\ \ ;\beta} &= 0, \quad T^{\alpha\beta} = (\rho+p)U^{\alpha}U^{\beta} + pg^{\alpha\beta} \\ T^{r\beta}_{\ \ ;\beta} &= (\rho+p)U^{\beta}U^{r}_{\ ;\beta} + g^{rr}p_{,r} = 0 \\ &= (\rho+p)U^{\beta}U^{\lambda}\Gamma^{r}_{\ \lambda\beta} + e^{-2\Lambda}p_{,r} = 0 \\ &= (\rho+p)(U^{0})^{2}\Gamma^{r}_{\ \ 00} + e^{-2\Lambda}p_{,r} = 0 \\ &= (\rho+p)(e^{-2\Phi})(e^{-2\Lambda}e^{2\Phi}\Phi_{,r}) + e^{-2\Lambda}p_{,r} = 0 \\ &- \frac{\mathrm{d}p}{\mathrm{d}r} = (\rho+p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} \end{split}$$

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