# Spherical solutions for stars

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General Relativity I Presentations
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# Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- real stars





Introduction

- model stars using spherical symmetry
- Schwarzschild metric
- T–O–V equation
- real stars

- I will model stars using GR assuming spherical symmetry
- I will derive the Schwarzschild metric and T–O–V equation
- finally I will look into specific types of stars

# Spherically symmetric coordinates





Spherically symmetric coordinates

- First we need to derive our coordinate system

# Two-sphere in flat spacetime

## General metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

## Metric on 2-sphere

$$\mathrm{d}l^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \equiv r^2\mathrm{d}\Omega^2$$

Schutz (2009, p. 256)





Two-sphere in flat spacetime
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Schutz (2009, p. 256)

- we start with the simplest spherically symmetric coordinates
  - flat spacetime
- 2-sphere in Minkowski spacetime
  - introduce $\mathrm{d}\Omega^2$ for compactness

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# Two-sphere in curved spacetime

**Metric on 2-sphere**

$$dl^2 = f(r', t) d\Omega^2$$

**Relation to $r$**

$$f(r', t) \equiv r^2$$

Schutz (2009, pp. 256–257)





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- inclusion of curvature makes $r^2$ some function of $r'$ and $t$

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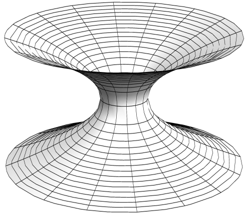
# Meaning of $r$



Mark Hannam

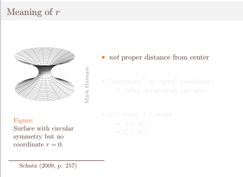
Figure:
Surface with circular symmetry but no coordinate $r = 0$.

- *not* proper distance from center
- "curvature" or "area" coordinate
  - radius of curvature and area
- $r = \text{const}$, $t = \text{const}$
  - $A = 4\pi r^2$
  - $C = 2\pi r$

Schutz (2009, p. 257)








- $r$ is not necessary the "distance from the center"
- it is merely a coordinate – "curvature" or "area" coordinate
- for instance, we may have a spacetime where the center is missing
  - example: Schwarzschild wormhole spacetime
- surface of constant $(r, t)$ is a two-sphere of area $A$ and circumference $C$

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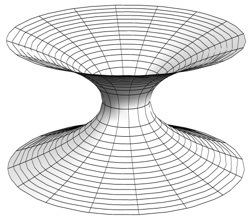

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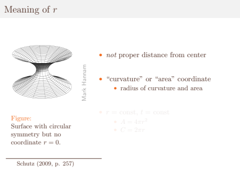

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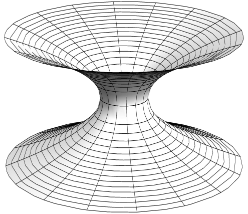

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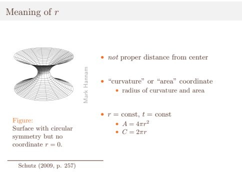

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# Spherically symmetric spacetime

**General metric**

$$ds^2 = g_{00}\,dt^2 + 2g_{0r}\,dr\,dt + g_{rr}\,dr^2 + r^2 d\Omega^2$$

$g_{00}$, $g_{0r}$, and $g_{rr}$: functions of $t$ and $r$

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- now consider not only surface of 2-sphere, but whole spacetime
- now we have some unknown $g_{00}$, $g_{rr}$, and cross term $g_{0r}$
- cross term $g_{0r}$
- cross terms $g_{0i}$ for $i \in \{\theta, \phi\}$ are zero from symmetry
- need more constraints to say anything particular about them

# Static spacetimes





Static spacetimes

- now I will impose the static constraint

# Motivation

- leads to simple derivation of Schwarzschild metric
- unique solution to spherically symmetric, asymptotically flat Einstein vacuum field equations (Birkhoff's theorem)

Schutz (2009, p. 263) and Misner, Thorne, and Wheeler (1973, p. 843)





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- we choose the constraint of a static spacetime because
  - it allows us to easily derive the Schwarzschild metric
  - according to Birkhoff's theorem, this metric is the unique solution to the Einstein vacuum field equations for spherically symmetric, asymptotically flat spacetimes
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# Definition

A spacetime is static if we can find a time coordinate $t$ for which

(i) the metric independent of $t$

$$g_{\alpha\beta,t} = 0$$

(ii) the geometry unchanged by time reversal

$$t \to -t$$

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- now I define "static"
- first condition is that the metric is independent of time
  - by itself, this condition is called "stationary"
- second condition is that metric unaffected by time reversal
- e.g. rotating stars are stationary but not static

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# Time reversal

$$\mathbf{\Lambda} : (t, x, y, z) \to (-t, x, y, z)$$

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} g_{\alpha\beta} = g_{\alpha\beta}$$

**Transformation**

$$\Lambda^{0}{}_{\bar{0}} = x^{0}{}_{,\bar{0}} = \frac{\partial t}{\partial(-t)} = -1$$

$$\Lambda^{i}{}_{\bar{i}} = x^{i}{}_{,\bar{i}} = \frac{\partial x^i}{\partial x^i} = 1$$

**Metric**

$$g_{\bar{0}\bar{0}} = (\Lambda^{0}{}_{\bar{0}})^2 g_{00} = g_{00}$$

$$g_{\bar{r}\bar{r}} = (\Lambda^{r}{}_{\bar{r}})^2 g_{rr} = g_{rr}$$

$$g_{\bar{0}\bar{r}} = \Lambda^{0}{}_{\bar{0}} \Lambda^{r}{}_{\bar{r}} g_{0r} = -g_{0r}$$

Schutz (2009, p. 258)





- now I use the static constraint to simplify the metric
- transformation
  - $(0, 0)$ term is $\mathrm{d}t/\mathrm{d}(-t)$
  - spatial terms are 1 if transformed to themselves
  - cross-terms are all zero, as coordinates independent of each other
- transformed metric
  - $(0, 0)$ term is unchanged, as $-1$ is squared
  - $(r, r)$ term is unchanged, as transformation is 1
  - $(0, r)$ term is negated, but must still be equal, so it's zero
    - no cross terms

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# The metric

## Simplified metric

$$ds^2 = g_{00}\,dt^2 + g_{rr}\,dr^2 + r^2 d\Omega^2$$

## Replacement

$$g_{00} \to -e^{2\Phi}, \quad g_{rr} \to e^{2\Lambda}, \quad \text{provided } g_{00} < 0 < g_{rr}$$

## Static spherically symmetric metric

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- we assume $g_{00}$ to be negative, and $g_{rr}$ to be positive
  - signature is $(-,+,+,+)$
  - holds inside stars but not black holes
- limits at infinity tell us that spacetime is *asymptotically flat*
  - $\Phi = \Lambda = 0 \implies e^{2\Phi} = e^{2\Lambda} = 1$ and $\mathbf{g} = \eta$

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# Einstein Tensor

## General Einstein tensor

$$G_{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

## Einstein tensor components

$$G_{00} = \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r} [r(1 - e^{-2\Lambda})]$$

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$$G_{\theta\theta} = r^2 e^{-2\Lambda} [\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

Schutz (2009, pp. 165, 260)





- now we can use the metric to derive the Riemann tensor
- from that the Einstein tensor
- the derivation is involved, so we will just take them as is
- we're going to use some of these components later on
- note that *prime* denotes d/d$r$

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# Static perfect fluid





Static perfect fluid

- stars are fluids – for simplicity we assume perfect
- thus we will impose additional constraints accordingly

# Four-velocity

## Constraints

$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

## Solving for $U^0$

$$g_{00} U^0 U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

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Schutz (2009, p. 260)





- static fluid, so in MCRF three-velocity components all zero
- we find the only non-zero term, $U^0$, by relating to the dot product
- lower it with the metric, to use in next part

$$\begin{aligned} g_{00} U^0 U^0 = -1 &\implies (U^0)^2 = (-g_{00})^{-1} \\ &\implies U^0 = (-g_{00})^{-1/2} \\ &\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi} \end{aligned}$$

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# Four-velocity

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$$U^i = 0 \text{ (static)} \qquad \vec{U} \cdot \vec{U} = -1 \text{ (conservation law)}$$

## Solving for $U^0$

$$g_{00} U^0 U^0 = -1 \implies U^0 = (-g_{00})^{-1/2} = e^{-\Phi}$$

## Solving for $U_0$

$$U_0 = g_{00} U^0 = -e^{\Phi}$$

Schutz (2009, p. 260)





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Schutz (2009, p. 260)

- static fluid, so in MCRF three-velocity components all zero
- we find the only non-zero term, $U^0$, by relating to the dot product
- lower it with the metric, to use in next part

$$\begin{aligned} g_{00} U^0 U^0 = -1 &\implies (U^0)^2 = (-g_{00})^{-1} \\ &\implies U^0 = (-g_{00})^{-1/2} \\ &\implies U^0 = (e^{2\Phi})^{-1/2} = e^{-\Phi} \end{aligned}$$

# Stress–energy tensor

## Stress-energy tensor for perfect fluid

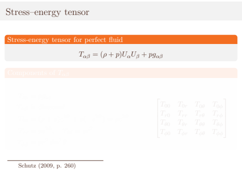
$$T_{\alpha\beta} = (\rho + p)U_\alpha U_\beta + p g_{\alpha\beta}$$

## Components of $T_{\alpha\beta}$

Schutz (2009, p. 260)







- $T_{i\alpha} = p g_{i\alpha}$ because spatial components of $U$ are zero
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# Equation of state

**Local thermodynamic equilibrium**

$$p = p(\rho, S) \approx p(\rho)$$

- pressure related to energy density and specific entropy
- we often deal with negligibly small entropies

Schutz (2009, p. 261)





Equation of state
Local thermodynamic equilibrium
$p = p(\rho, S) \approx p(\rho)$
- pressure related to energy density and specific entropy
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Schutz (2009, p. 261)

- in a static fluid we have local thermodynamic equilibrium
- pressure a function of density and specific entropy
- specific entropy assumed negligibly small

# Equations of motion

## Conservation of 4-momentum

$$T^{\alpha\beta}{}_{;\beta} = 0$$

- symmetries make only non-trivial solution $\alpha = r$

## Equation of motion

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r}$$

Schutz (2009, pp. 175, 261)





- first equation follows from conservation of 4-momentum
- due to symmetry, the only non-trivial solution is for $\alpha = r$
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# Mass function

## Einstein field equations

$$G_{00} = 8\pi T_{00} \implies \frac{1}{r^2} e^{2\Phi} \frac{\mathrm{d}}{\mathrm{d}r}[r(1 - e^{-2\Lambda})] = 8\pi \rho e^{2\Phi}$$

## $m(r)$

$$m(r) \equiv \frac{1}{2} r(1 - e^{-2\Lambda}) \quad \text{or} \quad g_{rr} = e^{2\Lambda} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

## Relation to energy density

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

Schutz (2009, pp. 260–262)





- inspect $(0,0)$ component of Einstein equations
- define the mass function, $m(r)$
- in Newtonian limit, $m(r)$ is mass within radius $r$

$$m(r) = 4\pi \int_0^r (r')^2 \rho(r') \, \mathrm{d}r'$$

- doesn't work in GR, because total energy is not localizable

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# $\Phi(r)$

## Einstein field equations

$$G_{rr} = 8\pi T_{rr} \implies -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\Phi' = 8\pi p e^{2\Lambda}$$

## $\Phi(r)$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

Schutz (2009, pp. 260–262)





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# Exterior Geometry





Exterior Geometry

- until now, we've not considered whether we were inside or outside star
- properties inside different than outside (obviously)
- we're going to inspect both cases, starting with outside

# Schwarzschild metric I

## Condition

$$\rho = p = 0$$

Schutz (2009, pp. 262–263)





- the external conditions just state we are in a vacuum
  - breaks down when matter surrounds star
- $m(r)$ is constant, we call it $M$
- $\mathrm{d}\Phi/\mathrm{d}r$ simplifies, and we can now integrate it to find $\Phi(r)$

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$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r-2M)} \qquad \Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right)$$

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$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho = 0 \qquad m(r) \equiv M$$

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]} = \frac{M}{r(r - 2M)} \qquad \Phi(r) = \frac{1}{2} \log\left(1 - \frac{2M}{r}\right)$$

Schutz (2009, pp. 262–263)





Schwarzschild metric I
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Schutz (2009, pp. 262–263)

- the external conditions just state we are in a vacuum
  - breaks down when matter surrounds star
- $m(r)$ is constant, we call it $M$
- $\mathrm{d}\Phi/\mathrm{d}r$ simplifies, and we can now integrate it to find $\Phi(r)$

# Schwarzschild metric II

## First two metric components

$$g_{rr} = e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad g_{00} = -e^{2\Phi} = -\left(1 - \frac{2M}{r}\right)$$

## Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

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- recall $g_{rr}$ from earlier
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- we have found the Schwarzschild metric!

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**Condition**

$$r \gg M$$

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**Far-field Schwarzschild metric (Cartesian)**

$$\mathrm{d}s^2 \approx -\left(1 - \frac{2M}{R}\right)\mathrm{d}t^2 + \left(1 + \frac{2M}{R}\right)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

$$R^2 \equiv x^2 + y^2 + z^2$$

Schutz (2009, pp. 263)





- far-field metric of a star (far away)
- consider $m(r)$ to be total mass $M$
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# Interior structure





Interior structure

- now we look at the remaining, and most interesting regime
  - inside the star
- our assumptions from outside the star no longer hold

# Tolman–Oppenheimer–Volkov (T–O–V) equation

**Condition**

$$\rho \neq 0 \quad p \neq 0$$

**Recall**

$$(\rho + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

**T–O–V equation**

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Schutz (2009, pp. 261–264)





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- revisit two earlier equations
- substitute one into the other
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- gives us an ODE relating
  - pressure $p$
  - density $\rho$
  - mass function $m(r)$
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    - eventually hope to solve all quantities in terms of $r$

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# System of coupled differential equations

## T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}$$

## Mass function

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho$$

## Equation of state

$$p = p(\rho)$$

Schutz (2009, pp. 261–262, 264)





System of coupled differential equations
T–O–V equation
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Equation of state
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Schutz (2009, pp. 261–262, 264)

- T–O–V equation coupled with $\mathrm{d}m/\mathrm{d}r$ and $p(\rho)$
  - 3 equations
  - 3 unknowns ($m$, $\rho$, $p$)
  - $\Phi(r)$ only intermediate variable
- can integrate to find $m(r)$, $\rho(r)$, and $p(r)$

# Newtonian hydrostatic equilibrium

**Newtonian limit**

$$p \ll \rho; \quad 4\pi r^3 p \ll m; \quad m \ll r$$

**Equation of hydrostatic equilibrium**

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} = -\frac{\rho m(r)}{r^2}$$

Schutz (2009, pp. 265–266) and Hansen and Kawaler (1994, p. 3)





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# Constant density solution I

## Constraint

$$\rho(r) \equiv \rho_0$$

## Mass function

$$m(r) = \frac{4}{3}\pi\rho_0 \begin{cases} r^3, & r \leq R, \\ R^3, & r \geq R. \end{cases}$$

Schutz (2009, pp. 266-267)





- because it is the simplest case, we are going to investigate a star of uniform density, $\rho(r) \equiv \rho_0$
  - this is unphysical
  - for instance, the speed of sound in such a star is infinite
  - neutron star density is *almost* uniform
  - also leads us to a result which holds for all stellar densities
- easy to obtain mass function from earlier differential equation
  - equal to the density times the volume of the sphere enclosed by radius $r$ inside
  - equal to the density times the volume of the entire star $(r = R)$ when outside
  - continuous at the boundary

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## T–O–V equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} = -\frac{4}{3}\pi r \frac{(\rho_0 + p)(\rho_0 + 3p)}{1 - \frac{8}{3} r^2 \rho_0}$$

## Integrated from center to internal radius $r$

$$\frac{\rho_0 + 3p}{\rho_0 + p} = \frac{\rho_0 + 3p_c}{\rho_0 + p_c}\sqrt{1 - 2m/r}$$

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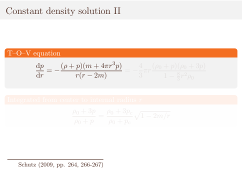

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- we can substitute $m(r)$ for $r \leq R$, to simplify it as shown
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- we integrate the differential equation from the center ($r = 0$, $p = p_c$) to some radius ($r = r$, $p = p$)
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# Constant density solution III

## Radius $R$

$$R^2 = \frac{3}{8\pi\rho_0}\left[1 - \left(\frac{\rho_0 + p_c}{\rho_0 + 3p_c}\right)^2\right]$$

## Central pressure $p_c$

$$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$$

## Limit on $M/R$

$$M/R \to 4/9 \implies p_c \to \infty$$

Schutz (2009, pp. 266-267, 269)



2015-12-14



Constant density solution III

- at the surface, $r = R$ and $p = 0$
- can solve the previous equation for $R$
- from this, we can solve for $p_c$
  - this gives us an expression for the central pressure necessary
- we can see that this blows up when $M/R = 4/9$

$$3\sqrt{1 - 8/9} - 1 = 3\sqrt{1/9} - 1 = 1 - 0 = 0$$

- radius cannot be smaller than $(9/4)M$
  - less than the $2M$ needed for a black hole
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$R^2 = \frac{3}{8\pi\rho_0}\left[1 - \left(\frac{\rho_0 + p_c}{\rho_0 + 3p_c}\right)^2\right]$
Central pressure $p_c$
$p_c = \rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1}$
Limit on $M/R$
$M/R \to 4/9 \implies p_c \to \infty$
Schutz (2009, pp. 266-267, 269)

- at the surface, $r = R$ and $p = 0$
- can solve the previous equation for $R$
- from this, we can solve for $p_c$
  - this gives us an expression for the central pressure necessary
- we can see that this blows up when $M/R = 4/9$

$$3\sqrt{1 - 8/9} - 1 = 3\sqrt{1/9} - 1 = 1 - 0 = 0$$

- radius cannot be smaller than $(9/4)M$
  - less than the $2M$ needed for a black hole
- Buchdahl's theorem states that this is true in general for all stars
  - not just $\rho(r) \equiv \rho_0$

# Buchdahl's theorem

- even for non-constant density, $M/R < 4/9$
- intuitive explanation:

Carroll (2004, pp. 234)





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- restate $M/R < 4/9$ from Buchdahl's theorem
- give Carroll's intuitive explanation
  - if we assume there is a maximum sustainable density in nature
  - and we consider an object which fills a sphere with radius $R$
  - then the most massive possible object within that volume would have a uniform density
  - all other objects would need to have a lower density

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- intuitive explanation:
  - assume there *is* a maximum sustainable density, $(M/R)_{\max}$
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  - most massive possible object would have maximum density everywhere
  - all other sustainable objects have a lower $M/R$

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# Realistic stars





Realistic stars

- now we're going to have a brief overview of real stars

# White dwarfs

- end-of-life for low mass stars
- held up by electron degeneracy pressure
- Newtonian structure accurate to 1%

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{\rho m}{r^2}$$

- relativistic effects important on stability and pulsation for

$$10^8 \mathrm{g\,cm^{-3}} \lesssim \rho_c \lesssim 10^{8.4} \mathrm{g\,cm^{-3}}$$

Misner, Thorne, and Wheeler (1973, p. 627)





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- end-of-life form of lower mass stars like our Sun is as a white dwarf
- core left over after a star loses its outer shell as a planetary nebula
- nuclear fusion has halted, and only pressure of degenerate electron gas supports them
  - Pauli exclusion principle
- structure can be described by the equation of HSE to high accuracy
- relativistic effects come into play for central densities:
  - over $10^8 \mathrm{g\,cm^{-3}}$
  - up until the maximum

# Neutron stars

- mass condensed further than white dwarf
- created in supernovae, or collapse of white dwarf

$$p^+ + e^- \to n^0 + \nu$$

- held up by neutron degeneracy pressure
- matter incredibly complex and possess many unknown properties

Schutz (2009, pp. 274–275)





Neutron stars
- mass condensed further than white dwarf
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$p^+ + e^- \to n^0 + \nu$
- held up by neutron degeneracy pressure
- matter incredibly complex and possess many unknown properties
Schutz (2009, pp. 274–275)

- when a star condenses beyond a white dwarf, it may become a neutron star
- occurs in the aftermath of a supernova, or collapse of white dwarf
- compression beyond neutron star would form a black hole
- kinetic energy of electrons high
  - allows energy release when combined with a proton
  - energy carried away by neutrino, and neutron left behind
- held up by neutron degeneracy pressure – Pauli again
- matter incredibly complex
  - suitable equation of state is a topic under active research

# Rotating stars

## Metric

$$ds^2 = -e^{2\nu}\,dt + e^{2\psi}(d\phi - \omega\,dt)^2 + e^{2\mu}(dr^2 + r^2\,d\theta^2),$$

$\nu$, $\psi$, $\omega$, and $\mu$: functions of $r$ and $\theta$

- stationary
- can still assume perfect fluid to high accuracy

Stergioulas (2003, p. 8)





Rotating stars
Metric
$ds^2 = -e^{2\nu}\,dt + e^{2\psi}(d\phi - \omega\,dt)^2 + e^{2\mu}(dr^2 + r^2\,d\theta^2)$,
$\nu$, $\psi$, $\omega$, and $\mu$: functions of $r$ and $\theta$
- stationary
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Stergioulas (2003, p. 8)

- much more complicated when we allow for rotation
- metric no longer static
  - addition of cross terms between $t$ and $\phi$
  - metric dependence on $\theta$ in addition to $r$
- metric is still stationary
- perfect fluid assumption works to high accuracy

# Pulsars

- rapidly rotating neutron stars
- magnetic field produces electromagnetic radiation
- pulses of radio waves observed with the right orientation
- introduction of strong magnetic field requires
  - consideration of coupled Einstein–Maxwell field equations
  - $T_{\alpha\beta}$ includes EM energy density – non-isotropic

Misner, Thorne, and Wheeler (1973, p. 628) and Stergioulas (2003, p. 28)





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Misner, Thorne, and Wheeler (1973, p. 628) and Stergioulas (2003, p. 28)

- pulsars are rapidly rotating neutron stars
- they have a strong magnetic field which causes emission of light
- magnetic poles may be offset from axis of rotation
- if observed from right angle, see pulses of radio light, like lighthouse
- by including a strong magnetic field, we need to
  - consider the coupled Einstein–Maxwell field equations, assuming
    - equilibrium
    - stationary
    - axisymmetric
    - internal electric current
  - need to include electromagnetic energy density to stress-energy tensor
  - this makes $T_{\alpha\beta}$ non-isotropic

# References





References

- You made it to the end!

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Spherical stars
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# Bonus slides





Bonus slides

# Equations of motion

$$
\begin{aligned}
T^{\alpha\beta}{}_{;\beta} &= 0, \quad T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta} \\
T^{r\beta}{}_{;\beta} &= (\rho + p) U^\beta U^r{}_{;\beta} + g^{rr} p_{,r} = 0 \\
&= (\rho + p) U^\beta U^\lambda \Gamma^r{}_{\lambda\beta} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p) (U^0)^2 \Gamma^r{}_{00} + e^{-2\Lambda} p_{,r} = 0 \\
&= (\rho + p) (e^{-2\Phi}) (e^{-2\Lambda} e^{2\Phi} \Phi_{,r}) + e^{-2\Lambda} p_{,r} = 0 \\
-\frac{\mathrm{d}p}{\mathrm{d}r} &= (\rho + p) \frac{\mathrm{d}\Phi}{\mathrm{d}r}
\end{aligned}
$$

Schutz (2009, pp. 101, 261)





Equations of motion
$T^{\alpha\beta}{}_{;\beta} = 0, \quad T^{\alpha\beta} = (\rho + p) U^\alpha U^\beta + p g^{\alpha\beta}$
$T^{r\beta}{}_{;\beta} = (\rho + p) U^\beta U^r{}_{;\beta} + g^{rr} p_{,r} = 0$
$= (\rho + p) U^\beta U^\lambda \Gamma^r{}_{\lambda\beta} + e^{-2\Lambda} p_{,r} = 0$
$= (\rho + p) (U^0)^2 \Gamma^r{}_{00} + e^{-2\Lambda} p_{,r} = 0$
$= (\rho + p) (e^{-2\Phi}) (e^{-2\Lambda} e^{2\Phi} \Phi_{,r}) + e^{-2\Lambda} p_{,r} = 0$
$-\frac{\mathrm{d}p}{\mathrm{d}r} = (\rho + p) \frac{\mathrm{d}\Phi}{\mathrm{d}r}$
Schutz (2009, pp. 101, 261)