Differential Equations Class Notes

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Contents

1 Introduction

Differential equations are equations that have one or more *derivatives* or *differentials*.

$$\frac{1}{x}\frac{dy}{dx} = \frac{1}{y} \quad \text{First order} \tag{1.1}$$

$$2x + y\,y'' = 5 \quad \text{Second order} \tag{1.2}$$

$$\tag{1.3}$$

$$y' = \frac{dy}{dx}$$

$$4\,\frac{d^2y}{dx^2} + xy = \sin x \tag{1.4}$$

What is a solution to a Differential Equation?

$$x\,\frac{dy}{dx} = 2y \tag{1.5}$$

Show that $y(x) = x^2$ is a solution.

$$\frac{dy}{dx} = 2x$$

$$x\,(2x) = 2(x^2)$$

$$2x^2 = 2x^2$$

For

$$(1 + xe^{xy})\,\frac{dy}{dx} + 1 + ye^{xy} = 0, \tag{1.6}$$

show that $x + y + e^{xy} = 0$ is a solution.

Implicit Solution

$$\begin{aligned}
\frac{d}{dx}\left(x+y+e^{xy}\right) &= \frac{d}{dx}(0) \\
1+\frac{dy}{dx}+e^{xy}\left(y+x\,\frac{dy}{dx}\right) &= 0 \\
1+\frac{dy}{dx}+y\,e^{xy}+x\,e^{xy}\frac{dy}{dx} &= 0 \\
\frac{dy}{dx}\left(1+x\,e^{xy}\right)+1+y\,e^{xy} &= 0
\end{aligned}$$

For

$$\frac{dy}{dx} = 2x, \tag{1.7}$$

show that the following are solutions

1. $y(x) = x^2$

$$\begin{aligned}
\frac{d}{dx}y(x) &= \frac{d}{dx}x^2 \\
\frac{dy}{dx} &= 2x
\end{aligned}$$

2. $y(x) = x^2 + 1$

$$\begin{aligned}
\frac{d}{dx}y(x) &= \frac{d}{dx}x^2 + 1 \\
\frac{dy}{dx} &= 2x
\end{aligned}$$

3. $y(x) = x^2 + C$

$$\frac{d}{dx}y(x) = \frac{d}{dx}x^2 + C$$
$$\frac{dy}{dx} = 2x$$

$$\frac{dp}{dt} = \frac{1}{2}p - 450 \tag{1.8}$$

Show that $p(t) = 900 + C\,e^{t/2}$ is a solution.

$$\begin{aligned}
\frac{dp}{dt} &= 0 + C\,e^{t/2}\,\frac{1}{2} \\
&= \frac{C}{2}\,e^{t/2} \\
\text{LHS} &= \frac{C}{2}\,e^{t/2} \\
\text{RHS} &= \frac{1}{2}\left(900 + C\,e^{t/2}\right) - 450 \\
&= \frac{C}{2}\,e^{t/2}
\end{aligned}$$

What if $p(0) = 850 \;\leftarrow$ (initial condition)

$$850 = 900 + C\,e^{0/2}$$

$$850 = 900 + C \Rightarrow C = -50$$

We now have an *Initial Value Problem* (IVP).

$$\begin{cases} \frac{dp}{dt} = \frac{1}{2}p - 450 \\ p(0) = 850 \end{cases}$$

Solution is

$$p(t) = 900 - 50\, e^{t/2}$$

$$\frac{dy}{dx} = f(x, y), \quad \frac{dy}{dx} \text{ is the slope.}$$

Example: Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to $1/2$ mice/month assuming no owls present. When owls are present, they eat the mice. Suppose that the owls eat on average 15 mice per day. Assuming there are 30 days in a month, write a differential equation to describe the above.

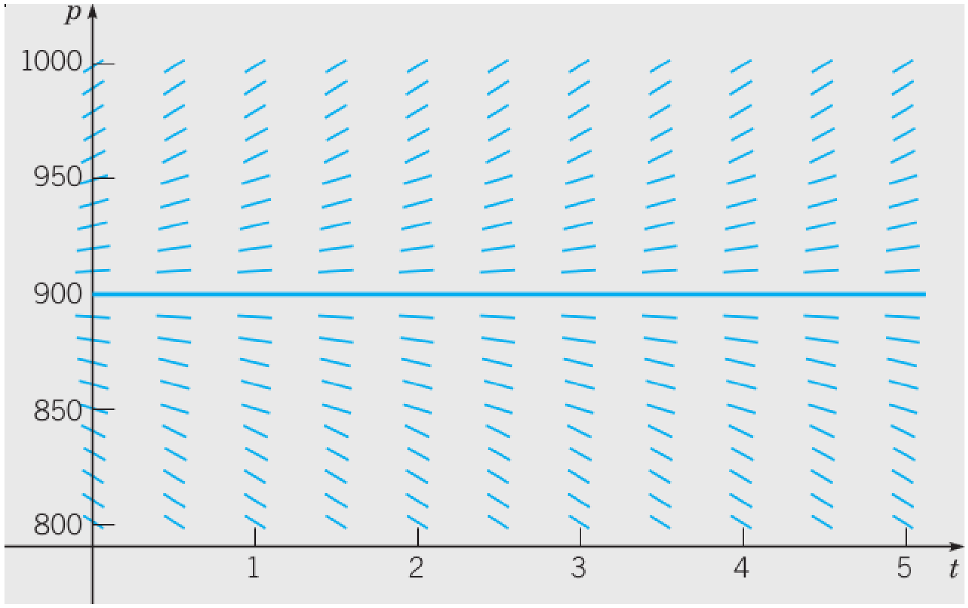
$$p = \text{mouse population}, \quad t = \text{time in months}$$

$$\frac{dp}{dt} \propto p$$
$$\frac{dp}{dt} = kp = \frac{1}{2}\, p - 450, \quad (450 \text{ is the mice eaten per month})$$

When $p = 900$, $\frac{dp}{dt} = 0$. This is the equilibrium solution.

When $p > 900$, $\frac{dp}{dt} > 0$

When $p < 900$, $\frac{dp}{dt} < 0$



Next time: Classifications of Diff Eqns & Seperable Diff Eqns.

2 Classification of Differential Equations

Ordinary vs. Partial Differential Equation (ODE vs. PDE)

$\rightarrow$ based on number of independent variables

$$\text{(dedpendent variable)} \rightarrow y = f(x \leftarrow \text{(independent variable)})$$

- If there is one independent variable $\rightarrow$ ODE
- If there is more than one independent variable $\rightarrow$ PDE

Example 2.1

$$\frac{dp}{dt} = \frac{1}{2}p - 450 \rightarrow p(t) \quad \textit{one independant variable} \tag{2.1}$$

Example 2.2

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial v(x,t)}{\partial t} \rightarrow \textit{PDE (Heat equation)} \tag{2.2}$$

Example 2.3

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{\partial^2 v(x,t)}{\partial t^2} \rightarrow \textit{PDE (Wave equation)} \tag{2.3}$$

Based on number of *unknowns*

$\rightarrow$ system of equations

$$\begin{aligned} \frac{dx}{dt} &= 4x + 3t \\ \frac{dy}{dt} &= 5x - 2y \end{aligned}$$

Order of a differential equation is the *highest derivative* that appears in the equation.

Equation 1	$y' + 3y = 0$	1^{st} order	linear
Equation 2	$y'' + 3y' - 2t = 0$	2^{nd} order	linear
Equation 3	$\frac{d^4y}{dt^4} - \frac{d^2y}{dt^2} + 1 = e^{2t}$	4^{th} order	linear
Equation 4	$u_{xx} + u_t = 0$	2^{nd} order	linear
Equation 5	$u_{xx} + u\,u_{yy} = \sin t$	2^{nd} order	non-linear
Equation 6	$u_{xx} + \sin(u)u_{yy} = \cos t$	2^{nd} order	non-linear
Equation 7	$u_{xx} + u_{yy} + \sin(u) = 0$	2^{nd} order	nonlinear
Equation 8	$u_{xx} + u_{yy} + \sin(t) = 0$	2^{nd} order	linear
Equation 9	$u_{xx} + u_{yy} + u = 0$	2^{nd} order	linear

Table 1: Examples of different order and linear vs. non-linear differential equations.

2.1 Linear vs. Non-Linear

A linear ODE is of the form:

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t) \tag{2.4}$$

This is similar to a polynomial, which take the form

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = f(x) \tag{2.5}$$

But instead of it being the independent variables raised to decreasing powers, it's the derivative of the independent variables with decreasing orders of the derivatives.

2.2 Seperable Equations

Consider a first order ODE

$$\frac{dy}{dx} = f(x, y) \ldots \tag{2.6}$$

If we can express Equation 2.6 in the form of

$$M(x)\,dx + N(y)\,dy = 0 \tag{2.7}$$

then Equation 2.6 is called a *seperable equation*.

$$\textbf{Solve} \quad \frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{2.8}$$

$$\begin{aligned} \int (1 - y^2)\,dy &= \int x^2\,dx \\ \int N(y)\,dy &= \int M(x)\,dx \\ y - \frac{y^3}{3} &= \frac{x^3}{3} + C \quad \text{(Implicit Solution)} \end{aligned}$$

$$\textbf{Solve} \quad \frac{dy}{dx} = \frac{3x^2 - 4x + 2}{2(y - 1)} \tag{2.9}$$

$$\int 2(y-1)\,dy = \int 3x^2 - 4x + 2\,dx$$
$$y^2 - 2y = x^3 - 2x^2 + 2x + C$$

Complete the square by adding 1 to both sides (note that C is still a constant, so the 1 disappears)

$$(y-1)^2 = x^3 + 2x^2 + 2x + C$$
$$y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + C}$$
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C} \quad \text{(Explicit Solution)}$$

If, also, $y(0) = -1$, solve for y

$$-1 = 1 \pm \sqrt{0^3 + 0^2 + 0 + C} = 1 \pm \sqrt{C}$$
$$-2 = \pm\sqrt{C}$$
$$C = (-2)^2 = 4$$
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$
$$-1 = 1 \pm \sqrt{4} = 1 \pm 2$$
$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

How about $y(0) = 3,\ C = 4$?

$$3 = 1 \pm \sqrt{4} = 1 \pm 2$$
$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

$$\textbf{Solve} \quad \begin{cases} y' &= \frac{4x-x^3}{4+y^3} \\ y(0) &= 1 \end{cases} \tag{2.10}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{4x-x^3}{4+y^3} \\
\int 4+y^3\,dy &= \int 4x-x^3\,dx \\
4y+\frac{1}{4}y^4 &= 2x^2-\frac{1}{4}x^4+C \\
4(1)+\frac{1}{4}1^4 &= 0-0+C \\
4+\frac{1}{4} &= C \\
C &= \frac{17}{4} \\
4y+\frac{1}{4}y^4 &= 2x^2-\frac{1}{4}x^4+\frac{17}{4}
\end{aligned}$$

2.3 Exact Differential Equations

Assume we have a solution

$$\Psi(x,y) = C \tag{2.11}$$

Then we will find what this solution is.

Theorem 2.4

$M(x,y)+N(x,y)\frac{dy}{dx}=0$ *is called exact if* $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$.

For Equation 2.11,

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0. \tag{2.12}$$

$$\frac{\partial \Psi}{\partial x} = M(x,y), \quad \frac{\partial \Psi}{\partial y} = N(x,y) \tag{2.13}$$

Example 2.5

$$
\begin{aligned}
2x + y^2 + 2xy\,\frac{dy}{dx} &= 0 \\
2x + y^2 &= M(x, y) \\
2xy &= N(x, y) \\
\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} &= 2y \\
\int \frac{\partial \Psi}{\partial x} &= \int \left(2x + y^2\right)\,dx \\
&= x^2 + xy^2 + c_1 \\
\int \frac{\partial \Psi}{\partial y} &= \int 2xy\,dy \\
&= xy^2 + c_2 \\
\Psi(x, y) &= x^2 + xy^2 + h(y) \\
\Psi(x, y) &= xy^2 + h(x) \\
\frac{\partial \Psi}{\partial y} &= 2xy + h'(y) \\
h'(y) &= 0 \\
\int h'(y)dy &= \int 0dy \\
h(y) &= c_3 \\
x^2 + xy^2 + c_3 &= c_4 \\
x^2 + xy^2 &= C
\end{aligned}
$$

Example 2.6

$$
\begin{aligned}
2xy + \left(x^2 + 1\right)\frac{dy}{dx} &= 0 \\
2xy &= M(x, y) \\
x^2 + 1 &= N(x, y)
\end{aligned}
$$

First we must see if they are exact

$$\begin{aligned}
\frac{\partial M}{\partial y} &= 2x \\
\frac{\partial N}{\partial x} &= 2x \\
\frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \\
x^2y + y &= C \\
\frac{\partial \Psi(x,y)}{\partial x} &= 2xy \\
\int \frac{\partial \Psi(x,y)}{\partial x} dx &= \int 2xy \, dx \\
\Psi(x,y) &= x^2y + h(y)
\end{aligned}$$

Now take the derivative with respect to y *and compare it to the known* $\frac{\partial \Psi}{\partial y}$ *to find* $h(y)$

$$\begin{aligned}
\frac{\partial \Psi}{\partial y} &= x^2 + h'(y) \\
\cancel{x^2} + h'(y) &= \cancel{x^2} + 1 \\
\int h'(y) \, dy &= \int 1 \, dy \\
h(y) &= y + c_1 \\
\Psi(x,y) &= x^2y + y + c_1 \\
x^2y + y &= C
\end{aligned}$$

Example 2.7

$$\begin{aligned}
\left(x^2 + y\right) + (x + \cos(y)) \frac{dy}{dx} &= 0 \\
M(x,y) &= x^2 + y \\
N(x,y) &= x + \cos(y) \\
\frac{\partial M}{\partial x} &= 2x \\
\frac{\partial N}{\partial x} &=
\end{aligned}$$

Example 2.8

$$2xy + (x^2 + 1)\frac{dy}{dx} = 0$$
$$\frac{\partial M}{\partial y} = 2x,\ \frac{\partial N}{\partial x} = 2x$$

Example 2.9

$$\begin{cases} x^2 \frac{dy}{dx} = x^3 - 2xy \\ y(1) = 3 \end{cases}$$

$$M(x,y) = -x^3 + 2xy$$

$$N(x,y) = x^2$$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

It is exact, so there must be a solution of the form:

$$\Psi(x,y) = C$$

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{\partial \Psi}{\partial x} = -x^3 + 2xy$$

$$\frac{\partial \Psi}{\partial y} = x^2$$

Integrate:

$$\int \frac{\partial \Psi}{\partial y} = \int x^2 \, dy$$

$$\Psi(x,y) = x^2 y + h(x)$$

$$\frac{\partial \Psi}{\partial x} = 2xy + h'(x)$$

$$-x^3 + 2xy = 2xy + h'(x)$$

$$h'(x) = -x^3$$

$$h(x) = \int -x^3 \, dx$$

$$= -\frac{x^4}{4} + C$$

$$\Psi(x,y) = x^2 y - \frac{x^4}{4} + C$$

$$x^2 y - \frac{x^4}{4} = C$$

$$
\begin{aligned}
y(1) &= 3 \\
1^2 \cdot 3 - \frac{1^4}{4} &= C \\
3 - \frac{1}{4} &= C \\
C &= {}^{11}\!/\!_4 \\
x^2 y - \frac{x^4}{4} &= {}^{11}\!/\!_4
\end{aligned}
$$

Example 2.10

$$
\begin{aligned}
(x^2 + y) + (x - \sin(y))\frac{dy}{dx} &= 0 \\
M = (x^2 + y),\ N &= (x - \sin(y)) \\
\frac{\partial M}{\partial y} &= 1 \\
\frac{\partial N}{\partial x} &= 1 \\
&\textit{It is exact}
\end{aligned}
$$

2.4 Integrating Factor

Example 2.11

$$
\begin{aligned}
\frac{x^2 + y}{x} + \frac{x - \sin(y)}{x}\frac{dy}{dx} &= 0 \\
M = \frac{x^2 + y}{x},\ N &= \frac{x - \sin(y)}{x} \\
\frac{\partial M}{\partial y} &= {}^1\!/\!_x \\
\frac{\partial N}{\partial x} &= \frac{\sin(y)}{x^2} \\
&\textit{It is not exact}
\end{aligned}
$$

Both examples have the same solution, but one is exact and the other is not. Example 2.11 is simply Example 2.10 divided by x.

- "Key term" (x) has been divided out!
- Key term $\rightarrow$ Integrating factor.

$$y' + p(x)y = g(x) \tag{2.14}$$

The Integrating factor is

$$\begin{aligned}
\mu(x) &= e^{\int p(x)\,dx} \\
\mu'(x) &= e^{\int p(x)\,dx} \cdot \frac{d}{dx} \int p(x)\,dx \\
\mu'(x) &= e^{\int p(x)\,dx} \cdot p(x) \\
\mu'(x) &= \mu(x)\,p(x) \\
\mu(x)\,y' + \mu(x)\,p(x)\,y &= \mu(x)\,g(x) \\
\mu(x)\,y' + \mu'(x)\,y &= \mu(x)\,g(x) \\
\int \frac{d}{dx}\,(\mu(x)\,y) &= \int \mu(x)\,g(x)\,dx \\
\mu(x)\,y &= \int \mu(x)\,g(x)\,dx \\
y &= \frac{1}{\mu(x)} \int \mu(x)\,g(x)\,dx
\end{aligned}$$

Example 2.12

$$\begin{aligned}
&\textit{Solve } y' + 2y = e^x \\
&p(x) = 2,\ g(x) = e^x \\
\mu(x) &= e^{\int 2\,dx} = e^{2x} \\
y &= \frac{1}{e^{2x}} \int e^{2x}\,e^x\,dx \\
&= e^{-2x} \int e^{3x}\,dx \\
&= e^{-2x} \left(\frac{1}{3}\,e^{3x} + c \right) \\
&= \frac{1}{3}\,e^x + c\,e^{-2x}
\end{aligned}$$

Example 2.13

$$\begin{aligned}
\frac{dy}{dx} &= 4 - \frac{2y}{x} \\
\frac{dy}{dx} + \frac{2y}{x} &= 4 \\
p(x) = \frac{2}{x},&\ g(x) = 4 \\
\mu(x) &= e^{\int p(x)\,dx} \\
&= e^{2\ln x} \\
\ln(a^b) &= b\ln(a) \\
&= e^{\ln x^2} \\
&= x^2 \\
y &= \frac{1}{x^2}\int 4\,x^2 \\
&= x^{-2}\left(\frac{4}{3}\,x^3 + c\right) \\
y(x) &= \frac{4}{3}\,x + c\,x^{-2}
\end{aligned}$$

Example 2.14

$$\begin{cases} \frac{dy}{dx} + 2xy = 1, \\ y(1) = 2 \end{cases}$$

$$\begin{aligned}
p(x) = 2x,&\ g(x) = 1 \\
\mu(x) &= e^{\int p(x)\,dx} \\
&= e^{x^2} \\
y &= \frac{1}{e^{x^2}}\int e^{x^2}\,dx
\end{aligned}$$

You can't integrate e^{x^2}

Example 2.15

$$x^2y^3 + x(1+y^2)y' = 0$$

$$M_y = 3x^2y^2,\ N_x = 1+y^2$$

$$M_y \neq N_x \ \textit{Not exact}$$

$$\mu(x,y) = \frac{1}{xy^3}$$

$$\frac{x^2y^3}{xy^3} + \frac{x(1+y^2)}{xy^3}y' = 0$$

$$x + \left(\frac{1}{y^3} + \frac{1}{y}\right)y' = 0$$

$$M_y = 0,\ N_x = 0$$

$$M_y = N_x \ \textit{Exact!}$$

$$\int M\,dx = \int x\,dx = \frac{x^2}{2} + C$$

$$\int N\,dy = \int \left(y^{-3} + y^{-1}\right)dy$$

$$= -\frac{1}{2y^2} + \ln|y| + C$$

$$\frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| = C$$

Theorem 2.16

$$\textit{If } \frac{N_x - M_y}{M} = Q(y),\ \textit{then}$$

$$M + Ny' = 0$$

$$\textit{has an integrating factor } \mu(y) = e^{\int Q(y)\,dy}$$

$$
\begin{aligned}
\mu M + \mu N y' &= 0 \\
e^{\int Q(y)\,dy} M + e^{\int Q(y)\,dy} N y' &= 0 \\
\tilde{M}_y &= e^{\int Q(y)\,dy} M \\
\tilde{N}_x &= e^{\int Q(y)\,dy} N \\
\tilde{M}_y &= \tilde{N}_x \\
\tilde{M}_y &= \frac{\partial}{\partial y}\left(e^{\int Q(y)\,dy} M\right) \\
&= e^{\int Q(y)\,dy} \cdot Q(y) M + e^{\int Q(y)\,dy} \cdot M_y \\
\tilde{N}_x &= \frac{\partial}{\partial x}\left(e^{\int Q(y)\,dy} N\right) \\
&= e^{\int Q(y)\,dy} \cdot N_x \\
N_x - M_y &= MQ(y) \\
N_x &= M_y + MQ(y) \\
\tilde{N}_x &= e^{\int Q(y)\,dy} \left(M_y + MQ(y)\right) \\
&= e^{\int Q(y)\,dy} \cdot M_y + MQ(y) e^{\int Q(y)\,dy} = \tilde{M}_y
\end{aligned}
$$

$\tilde{M}_y = \tilde{N}_x$, Exact!

Example 2.17

$$
\begin{aligned}
y + \left(2xy - e^{-2y}\right) y' &= 0 \\
N(x,y) &= 2xy - e^{-2y} \\
M(x,y) &= y \\
N_x = 2y,\ M_y &= 1
\end{aligned}
$$

Using Theorem 2.16:

$$
\begin{aligned}
\frac{N_x - M_y}{M} &= \frac{2y-1}{y} = 2 - \frac{1}{y} \\
Q(y) &= 2 - \frac{1}{y} \\
\mu(y) &= e^{\int 2 - \frac{1}{y}\,dy} \\
&= e^{2y - \ln|y|} \\
&= e^{2y} \cdot e^{-\ln|y|} \\
&= e^{2y} \cdot e^{\ln|\frac{1}{y}|} \\
&= \frac{1}{y} e^{2y} \\
\frac{1}{y} e^{2y} \left[y + \left(2xy - e^{-2y}\right) y'\right] &= 0 \\
e^{2y} + \frac{1}{y} e^{2y} \left(2xy - e^{-2y}\right) y' &= 0 \\
e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right) y' &= 0 \\
M &= e^{2y} \\
N &= \left(2xe^{2y} - \frac{1}{y}\right) \\
M_y &= 2e^{2y} \\
N_x &= 2e^{2y} \\
M_y = N_x &\ \textit{Exact!} \\
\int M\,dx &= xe^{2y} \\
\int N\,dy &= xe^{2y} - \ln|y| \\
xe^{2y} - \ln|y| &= C
\end{aligned}
$$

2.5 Modeling with First Order Equations

Example 2.18 *Mixing Problem*

At time $t = 0$*, a tank contains* Q_0 *lb of salt dissolved in 100 gal of water. Assume that water containing* $1/4$ *lb of salt/gal is entering the tank at a rate of* r *gal/min and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt* $Q(t)$ *in the tank at any time, and also find the limiting amount* Q_L *that is present after a very long time. If* $r = 3$ *and* $Q_0 = 2Q_L$*, find the time* T *after which the salt level is within* 2% *of* Q_L*. Also find the flow rate that is required if the value of* T *is not to exceed 45 min.*

Let $Q(t)$ *be the amount of salt at time* t

$$\frac{dQ}{dt} = Rate\ in - Rate\ out$$

$$Rate\ in = (Amount) \times (Rate)$$

$Amount = 1/4\ lb/gal$

$Rate = r\ gal/min$

Rate out:

$$\begin{cases} Amount: & \frac{Q\ [lb]}{100\ gal} \\ Rate: & r\ [gal/min] \end{cases}$$

$$\begin{aligned}
\frac{dQ}{dt} &= \left(\frac{1}{4}\,\frac{lb}{gal}\right)\left(r\,\frac{gal}{min}\right) - \left(\frac{Q}{100}\,\frac{lb}{gal}\right)\left(r\,\frac{gal}{min}\right) \\
&= \frac{1}{4}r - \frac{Q}{100}r \\
\frac{dQ}{dt} + \frac{Q}{100}r &= \frac{1}{4}r \\
\mu(t) &= e^{\int \frac{r}{100}\,dt} \\
&= e^{\frac{rt}{100}} \\
Q(0) &= Q_0
\end{aligned}$$

Example 2.19 *Example 3 from Page 57 in the textbook.*

Let $Q(t)$ be the amount of chemical in the pond at any time t.

$$\begin{aligned}
\frac{dQ}{dt} &= Rate\ in - Rate\ out \\
&= (2+\sin(2t))\frac{\text{g}}{gal}\left(5\,\frac{gal}{year}\right) \\
&\quad - \frac{Q}{10}\,\frac{\text{g}}{gal}\left(5\,\frac{gal}{year}\right) \\
&= (10+5\sin(2t))\,\frac{\text{g}}{year} - \frac{Q}{2}\,\frac{\text{g}}{year} \\
&= 10+5\sin(2t) - \frac{Q}{2} \\
Q(0) &= 0 \\
p(t) &= \frac{1}{2} \\
g(t) &= 10+5\sin(2t) \\
\mu(t) &= e^{\int p(t)\,dt} = e^{t/2} \\
Q &= e^{-t/2}\int e^{t/2}\cdot(10+5\sin(2t))\,dt \\
&= e^{-t/2}\left[10\int e^{t/2}\,dt + 5\int e^{t/2}\sin(2t)\,dt\right]
\end{aligned}$$

Integration by parts:

$$
\begin{aligned}
f(t) &= \int e^{t/2} \sin(2t)\, dt \\
&\begin{cases} v = e^{t/2} & dv = \sin(2t)\, dt \\ dv = {}^1\!/\!_2\, e^{t/2}\, dt & v = -{}^1\!/\!_2 \cos(2t) \end{cases} \\
&= -{}^1\!/\!_2\, e^{t/2} \cos(2t) + {}^1\!/\!_4 \int e^{t/2} \cos(2t)\, dt \\
g(t) &= {}^1\!/\!_4 \int e^{t/2} \cos(2t)\, dt \\
&\begin{cases} v = e^{t/2} & dv = \cos(2t)\, dt \\ du = {}^1\!/\!_2\, e^{t/2} & v = {}^1\!/\!_2 \sin(2t) \end{cases} \\
&= {}^1\!/\!_4 \left[{}^1\!/\!_2\, e^{t/2} \sin(2t) - {}^1\!/\!_4 \int e^{t/2} \sin(2t)\, dt \right] \\
I(t) &= \int e^{t/2} \sin(2t)\, dt \\
&\begin{cases} v = e^{t/2} & dv = \sin(2t)\, dt \\ du = {}^1\!/\!_2\, e^{t/2} & v = -{}^1\!/\!_2 \cos(2t) \end{cases} \\
&= -{}^1\!/\!_2\, e^{t/2} \cos(2t) + {}^1\!/\!_4 \int e^{t/2} \cos(2t)\, dt \\
&= -{}^1\!/\!_2\, e^{t/2} \cos(2t) + {}^1\!/\!_8\, e^{t/2} \sin(2t) - \frac{I}{16} \\
\frac{17}{16} I &= -{}^1\!/\!_2\, e^{-t/2} \cos(2t) + {}^1\!/\!_8\, e^{t/2} \sin(2t) + e
\end{aligned}
$$

Now we can substitute $I(t)$ back into $g(t)$ and then $f(t)$ to get our final solution.

Example 2.20 *Page 59, Problem 3*

A tank originally contains 100 gal of fresh water. Then water containing ½ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

$$
\begin{aligned}
\frac{d\tilde{Q}}{dt} &= Rate\ in - Rate\ out \\
\tilde{Q}(0) &= 0 \\
\frac{d\tilde{Q}}{dt} &= 1\frac{lb}{\text{min}} - \frac{Q}{50}\frac{lb}{\text{min}} \\
\frac{d\tilde{Q}}{dt} &= 1 - \frac{Q}{50} \\
1 &= \frac{d\tilde{Q}}{dt} + \frac{Q}{50} \\
\mu(t) &= e^{\int {}^1\!/_{50}\, dt} = e^{{}^t\!/_{50}} \\
Q(t) &= e^{-{}^t\!/_{50}} \int e^{{}^t\!/_{50}} \cdot 1\, dt \\
&= e^{-{}^t\!/_{50}} \left(50\, e^{{}^t\!/_{50}} + C\right) \\
&= 50 + Ce^{-{}^t\!/_{50}} \\
Q(0) &= 0 = 50 + C \\
C &= -50 \\
Q(t) &= 50 - 50e^{-{}^t\!/_{50}} \\
Q(10) &= 50 - 50e^{-{}^{10}\!/_{50}} \\
&\approx 9.06\ lb
\end{aligned}
$$

2.6 Bernoulli's Equation

Bernoulli's Equation is of the form

$$
\frac{dy}{dx} + p(x)y = g(x)y^n \tag{2.15}
$$

This is a more general case of the form we've used for integrating factors, where n was equal to 0

$$
\frac{dy}{dx} + p(x)y = g(x)\cancel{y^{0}}{}^{\nearrow 1}
$$

$$\text{Let } y = v^{\frac{1}{1-n}};\ y^n = v^{\frac{n}{1-n}}$$

$$\frac{dy}{dx} = \frac{v^{\frac{1}{1-n}-1}}{1-n}\frac{dv}{dx} = \frac{v^{\frac{n}{1-n}}}{1-n}\frac{dv}{dx}$$

$$\frac{v^{\frac{n}{1-n}}}{1-n}\frac{dv}{dx} + p(x)v^{\frac{1}{1-n}} = g(x)v^{\frac{n}{1-n}}$$

$$\text{Now divide everything by } \frac{n}{1-n} \left\{ \frac{v^{\frac{1}{1-n}}}{v^{\frac{n}{1-n}}} = v^{\frac{1}{1-n}-\frac{n}{1-n}} = v^{\frac{1-n}{1-n}} = v^1 = v \right.$$

$$\frac{dv}{dx} + (1-n)p(x)v = g(x)(1-n)$$

$$\frac{dy}{dx} + p(x)y = g(x)y^n$$

Example 2.21

$$\textit{Solve } \frac{dy}{dx} + 3y = e^x y^2$$

This is a Bernoulli with $n = 2$

$$y = v^{\frac{1}{1-2}} = v^{-1} \Rightarrow y = \frac{1}{v}$$

$$\begin{aligned}
\frac{dv}{dx} + (1-2)p(x)v &= g(x)(1-2) \\
\frac{dv}{dx} - p(x)v &= -g(x) \\
\frac{dv}{dx} - 3v &= -e^x \\
\mu(x) &= e^{\int -3\,dx} = e^{-3x} \\
v(x) &= e^{3x}\int e^{-3x}\cdot(-1)e^x\,dx \\
&= -e^{3x}\int e^{-3x}e^x\,dx \\
&= -e^{3x}\int e^{-2x}\,dx \\
&= -e^{3x}\left(-\frac{e^{-2x}}{2} + c_1\right) \\
&= \frac{e^x}{2} - c_1e^{3x} \\
&= \frac{e^x}{2} + Ce^{3x}
\end{aligned}$$

$$\text{but } y = \frac{1}{v} = v^{-1}$$

$$\text{so } y(x) = \left(\frac{e^x}{2} + Ce^{3x}\right)^{-1}$$

$$y = 0 \text{ is a solution as well}$$

2.7 Homogenous Equations

wrote it in my notebook

2.8 Second Order Linear Equations

$$\text{of the form } \frac{d^2y}{dx^2} = f(x, y, y') \text{ or } p(x)y'' + q(x)y' + r(x)y = g(x)$$

Example 2.22 *Solve $y'' - y = 0$ or $y'' = y$.*

$$\begin{cases} y_1(x) = e^x, \quad y_2(x) = e^{-x} \end{cases}$$

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x + c_2 e^{-x} \\ y'(x) &= c_1 e^x - c_2 e^{-x} \\ y''(x) &= c_1 e^x + c_2 e^{-x} = y(x) \end{aligned}$$

So $y(x) = c_1 e^x + c_2 e^{-x}$ is the general solution.

If we have $ay''+by'+cy=0$ where a, b, and c are constants, then $y=e^{mx}$ is a solution if $y'=me^{mx}$, $y''=m^2e^{mx}$

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$
$$e^{mx}\left[am^2+bm+c\right]=0$$

So $am^2+bm+c=0$ (characteristic equation)

There are three cases for characteristic equations

1. Two distinct roots $r_1\neq r_2$

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x}$$

2. Complex roots $m=\alpha\pm\beta i$

$$y(x)=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x)$$

3. Repeated roots $r_1=r_2=r$

$$y(x)=c_1e^{rx}+c_2xe^{rx}$$

A second order linear equation

$$y''+p(x)y'+q(x)y=g(x)$$

is called homogenous if $g(x)=0$ and inhomogenous if $g(x)\neq 0$.

If r_1 and r_2 are two distinct roots of a characteristic equation, then the general solution is given by

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x}$$

Example 2.23 *Solve* $y'' + 3y' - 10y = 0$

$$e^{mx}\left[m^2 + 3m - 10\right] = 0$$

$$m^2 + 3m - 10 = 0$$

$$(m+5)(m-2) = 0$$

$$m = -5,\ m = 2$$

$$y(x) = c_1 e^{-5x} + c_2 e^{2x}$$

For an IVP you would need $y(x_0) = y_0$ *and* $y'(x_0) = \tilde{y}_0$

Example 2.24

$$I.V.P.\ \begin{cases} y'' + 4y' + 3y = 0 \\ y(0) = 1, \qquad\qquad y'(0) = 1 \end{cases}$$

$$m^2 + 4m + 3 = 0$$

$$(x+1)(x+3) = 0$$

$$y(x) = c_1 e^{-x} + c_2 e^{-3x}$$

$$y'(x) = -c_1 e^{-x} - 3c_2 e^{-3x}$$

$$y(0) = 1 = c_1 + c_2$$

$$y'(0) = 1 = -c_1 - 3c_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$y(x) = 2e^{-x} - e^{-3x}$$

Example 2.25 *Solve* $x^2 + 2x + 2 = 0$

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} \\ &= -1 \pm i \end{aligned}$$

Example 2.26 *Solve* $y'' + 2y' + 2y = 0$

$$\begin{aligned} m^2 + 2m + 2 &= 0 \\ m &= -1 \pm i \\ r_1 &= -1 + i \\ r_2 &= -1 - i \\ y(x) &= c_1 e^{(-1+i)x} + c_2 e^{(-1-i)x} \\ &= c_1 e^{-x+xi} + c_2 e^{-x-xi} \\ &= c_1 e^{-x} \left[\cos(x) + i\sin(x)\right] + c_2 e^{-x} \left[\cos(-x) + i\sin(-x)\right] \\ & \begin{cases} -\sin(\theta) = \sin(\theta) \\ \cos(\theta) = \cos(-\theta) \end{cases} \\ &= c_1 e^{-x}\cos(x) + ic_1 e^{-x}\sin(x) + c_2 e^{-x}\cos(x) - ic_2 e^{-x}\sin(x) \\ &= (c_1 + c_2)e^{-x}\cos(x) + (c_1 - c_2)ie^{-x}\sin(x) \\ A &= c_1 + c_2 \\ B &= i(c_1 - c_2) \\ y(x) &= Ae^{-x}\cos(x) + Be^{-x}\sin(x) \end{aligned}$$

So, if $\alpha \pm i\beta$ *are the roots, the general solution is*

$$y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$$

Example 2.27 *The roots are:* $m = -2 \pm 7i$

Then the general solution must be

$$y(x) = Ae^{-2x}\cos(7x) + Be^{-2x}\sin(7x) = e^{-2x}\left[A\cos(7x) + B\sin(7x)\right]$$

Example 2.28

$$\begin{cases} y'' + 14y' + 149y = 0 \\ y(0) = -10, \qquad\qquad y'(0) = 8 \end{cases}$$

Example 2.29

$$\begin{cases} y'' + 4y' + 13y = 0 \\ y(0) = 0, \qquad\qquad y'(0) = 7 \end{cases}$$

For repeated roots, $r_1 = r_2$, so $y(x) = e^{r_1 x} = e^{r_2 x}$

Example 2.30

$$\begin{cases} y'' + 4y' + 4y = 0 \\ m^2 + 4m + 4 = 0 \\ (m+2)^2 = 0, \qquad m = -2 \end{cases}$$

$$\begin{aligned} y(x) &= v(x)e^{-2x} \\ y'(x) &= v'(x)e^{-2x} - 2v(x)e^{-2x} \\ &= e^{-2x}\left[v'(x) - 2v(x)\right] \\ y''(x) &= v''(x)e^{-2x} - 2v'(x)e^{-2x} + 4v(x)e^{-2x} - 2v'(x)e^{-2x} \\ &= e^{-2x}\left[v''(x) - 4v'(x) + 4v(x)\right] \end{aligned}$$

$$e^{-2x}\left[v''(x) - 4v'(x) + 4v(x)\right] + 4e^{-2x}\left[v'(x) - 2v(x)\right] + 4e^{-2x}v(x) = 0$$

$$e^{-2x}\left[v''(x) - 4v'(x) + 4v(x) + 4v'(x) - 8v(x) + 4v(x)\right] = 0$$

$$e^{-2x}v''(x) = 0$$

$$\begin{aligned}
v' &= c_1 \\
v &= \int c_1\,dx = c_1x + c_2 \\
v(x) &= c_1x + c_2 \\
y(x) &= (c_1x + c_2)\,e^{-2x} \\
y(x) &= c_2e^{-2x} + c_1xe^{-2x} \\
y'(x) &= -2c_2e^{-2x} + c_1e^{-2x} - 2c_1xe^{-2x}
\end{aligned}$$

$$\begin{bmatrix} xe^{-2x} & e^{-2x} \\ e^{-2x} - 2xe^{-2x} & -2e^{-2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

If $r_1 = r_2 = r$, the general solution is $y(x) = c_1e^{rx} + c_2xe^{rx}$

Example 2.31

$$\begin{cases} 16y'' - 40y' + 25y = 0 \\ y(0) = 3, \qquad\qquad y'(0) = {}^{-9}/_{4} \end{cases}$$

$$\begin{aligned}
16m^2 - 40m + 25 &= 0 \\
(4m-5)^2 &= 0 \\
m &= {}^{5}/_{4}
\end{aligned}$$

$$
\begin{aligned}
y(x) &= c_1 e^{5x/4} + c_2 x e^{5x/4} \\
y'(x) &= {}^5\!/_4\, c_1 e^{5x/4} + c_2 e^{5x/4} + {}^5\!/_4\, c_2 x e^{5x/4} \\
y(0) &= 3 = c_1 + 0 \\
y'(0) &= {}^{-9}\!/_4 = {}^5\!/_4\, c_1 + c_2 + 0 \\
c_1 &= 3 \\
c_2 &= {}^{-9}\!/_4 - {}^{5\cdot 3}\!/_4 = -6 \\
y(x) &= 3e^{5x/4} - 6xe^{5x/4} \\
&= 3e^{5x/4}\,(1 - 2x)
\end{aligned}
$$

Example 2.32

$$
\begin{cases}
y'' + 14y' + 149y = 0 \\
y(0) = -10, \qquad\qquad y'(0) = 8
\end{cases}
$$

$$
\begin{aligned}
m^2 + 14m + 149 &= 0 \\
m &= -7 \pm 10i \\
y(x) &= c_1 e^{-7x}\cos(10x) + c_2 e^{-7x}\sin(10x) \\
y'(x) &= -7c_1 e^{-7x}\cos(10x) - 10c_1 e^{-7x}\sin(10x) - 7c_2 e^{-7x}\sin(10x) + 10c_2 e^{-7x}\cos(10x) \\
&= (-7c_1 + 10c_2)e^{-7x}\cos(10x) + (-10c_1 - 7c_2)e^{-7x}\sin(10x)
\end{aligned}
$$

Example 2.33

$$
\begin{cases}
y'' + 4y' + 13y = 0 \\
y(0) = 0, \qquad\qquad y'(0) = 7
\end{cases}
$$

$$
\begin{aligned}
m^2 + 4m + 13 &= 0 \\
m &= -2 \pm 3i \\
y(x) &= c_1 e^{-2x}\cos(3x) + c_2 e^{-2x}\sin(3x) \\
y'(x) &= -2c_1 e^{-2x}\cos(3x) - 3c_1 e^{-2x}\sin(3x) - 2c_2 e^{-2x}\sin(3x) + 3c_2 e^{-2x}\cos(3x) \\
y(0) &= 0 = c_1 \\
y'(0) &= 7 = -2c_1 + 3c_2
\end{aligned}
$$

$$
\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -2 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right]
$$

$$
y(x) = e^{-2x}\cos(3x) + 3e^{-2x}\sin(3x)
$$

MISSING NOTES GO HERE

3 Transformations

3.1 Laplace Transform

The Laplace transform of $f(t)$ is denoted by

$$
\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s) \tag{3.1}
$$

Example 3.1 *Compute* $\mathcal{L}\{e^{at}\}$

$$\begin{aligned}
F(s) &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty \\
&= \frac{1}{a-s}\left[e^{-\infty} - e^0\right],\ \ s > a \\
&= -\frac{1}{a-s} = \frac{1}{s-a},\ \ s > a
\end{aligned}$$

Example 3.2 *Compute* $\mathcal{L}\{\sin(at)\}$

$$\begin{aligned}
F(s) &= \int_0^\infty \sin(at)e^{-st}\,dt \\
IBP &\begin{cases} u = e^{-st}, & du = -se^{-st} \\ v = -\frac{\cos(at)}{a}, & dv = \sin(at)\,dt \end{cases} \\
F(s) &= -\frac{\cos(at)e^{-st}}{a} - \frac{s}{a}\int_0^\infty \cos(at)e^{-st}\,dt \\
&\begin{cases} u = e^{-st}, & du = -se^{-st}\,dt \\ v = \frac{\sin(at)}{a}, & dv = \cos(at)\,dt \end{cases} \\
F(s) &= -\frac{\cos(at)e^{-st}}{a} - \frac{s}{a}\left(\frac{\sin(at)e^{-st}}{a} + \frac{s}{a}\int_0^\infty \sin(at)e^{-st}\,dt\right) \\
&= -\frac{\cos(at)e^{-st}}{a} - \frac{s}{a}\left(\frac{\sin(at)e^{-st}}{a} + \frac{s}{a}F(s)\right) \\
&= -\frac{\cos(at)e^{-st}}{a} - \frac{s}{a^2}\sin(at)e^{-st} - \frac{s^2}{a^2}F(s) \\
F(s)\left(1 + \frac{s^2}{a^2}\right) &= -\frac{e^{-st}}{a}\left(\cos(at) + \frac{s}{a}\sin(at)\right) \\
F(s) &= \left[-\frac{e^{-st}}{a}\left(\cos(at) + \frac{s}{a}\sin(at)\right)\frac{a^2}{a^2+s^2}\right]_0^\infty \\
&= 0 - \left(-\frac{1}{a}(1+0)\right)\frac{a^2}{a^2+s^2},\ \ s > 0 \\
&= \frac{a}{a^2+s^2},\ \ s > 0
\end{aligned}$$

Properties

1. $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

2. $\mathcal{L}\{c\, f(t)\} = c\ \mathcal{L}\{f(t)\}$

Example 3.3 *Compute* $\mathcal{L}\{3\sin(2t)\}$

$$\begin{aligned}\mathcal{L}\{3\sin(2t)\} &= 3\,\mathcal{L}\{\sin(2t)\}\\ &= 3\,\frac{2}{s^2+4},\ \ s>0\\ &= \frac{6}{s^2+4},\ \ s>0\end{aligned}$$

Example 3.4 *Compute* $\mathcal{L}\{e^t - e^{2t}\}$

$$\begin{aligned}\mathcal{L}\{e^t - e^{2t}\} &= \mathcal{L}\{e^t\} - \mathcal{L}\{e^{2t}\}\\ &= \frac{1}{s-1} - \frac{1}{s-2},\ \ s>2\\ &= -\frac{1}{(s-2)(s-1)},\ \ s>2\end{aligned}$$

3.2 Inverse Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{f(t)\}\right\} = \mathcal{L}^{-1}\left\{F(s)\right\} \tag{3.2}$$

Properties

1. $\mathcal{L}^{-1}\{F(s)+G(s)\} = \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\}$

2. $\mathcal{L}^{-1}\{c\, F(s)\} = c\ \mathcal{L}^{-1}\{F(s)\}$

Example 3.5 *Compute* $\mathcal{L}^{-1}\left\{\frac{3}{s}\right\}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} &= 3\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\ &= 3(1) = 3\end{aligned}$$

Example 3.6

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} &= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} \\ &= 4e^{-t}\end{aligned}$$

Example 3.7

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s+2}\right\} &= 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} \\ \frac{A}{s+1}+\frac{B}{s+2} &= \frac{2}{(s+1)(s+2)} \\ A(s+2)+B(s+1) &= 2 \\ Set\ s &= -2 \\ B=-2,\ A &= 2 \\ f(s) &= 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} + (-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} \\ &= 2e^{-t} - 2e^{-2t}\end{aligned}$$

Do for homework:

Example 3.8

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\} \\
s^2+2s+5 &= s^2+2s+1+4 \\
&= (s+1)^2+4 \\
&= (s+1)^2+2^2 \\
f(t) &= \mathcal{L}^{-1}\left\{\frac{2s+4}{(s+1)^2+2^2}\right\} \\
& \textit{From the table:} \\
e^{at}\sin(bt) &= \mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\},\ s>a \\
e^{at}\cos(bt) &= \mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\},\ s>a \\
f(t) &= \mathcal{L}^{-1}\left\{\frac{2(s+1)+2}{(s+1)^2+2^2}\right\} \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}(a=-1,\ b=2) \\
&+\quad \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}(a=-1,\ b=2) \\
&= 2e^{-t}\cos(2t)+e^{-t}\sin(2t)
\end{aligned}
$$

Theorem 3.9

$$
\mathcal{L}\left\{f'(t)\right\} = s\,\mathcal{L}\left\{f(t)\right\} - f(0)
$$

Proof

$$
\begin{aligned}
\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty f'(t)e^{-st}\,dt \\
&\begin{cases} u = e^{-st}, \quad du = -se^{-st}\,dt \\ v = f(t), \quad dv = f'(t)\,dt \end{cases} \\
&= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st}f(t)\,dt \\
&= \left(\cancelto{0}{e^{-s\cdot\infty}f(\infty)} - e^{-s\cdot 0}f(0)\right) + s\int_0^\infty f(t)e^{-st}\,dt \\
&= 0 - f(0) + 5\,\mathcal{L}\left\{f(t)\right\}
\end{aligned}
$$

Example 3.10

$$\mathcal{L}\left\{f''(t)\right\} = \mathcal{L}\left\{[f'(t)]'\right\}$$

$$
\begin{aligned}
&= s\,\mathcal{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(s\,\mathcal{L}\left\{f(t)\right\} - f(0)\right) - f'(0) \\
&= s^2\,\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)
\end{aligned}
$$

Example 3.11

$$
\begin{aligned}
\mathcal{L}\{f'''(t)\} &= \mathcal{L}\{[f''(t)]'\} \\
&= s\,\mathcal{L}\{f''(t)\} - f''(0) \\
&= s\left[s^2\,\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)\right] - f''(0) \\
&= s^3\,\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)
\end{aligned}
$$

1. $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

2. $\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

3. $\mathcal{L}\{f'''(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$

4. $\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

$= s^n\,\mathcal{L}\{f(t) + \sum_{i=1}^{n} -s^{n-i}f^{i-1}(0)$

Example 3.12

$$Solve\ \begin{cases} y' = y \\ \\ y(0) = 1 \end{cases} \quad or\ \begin{cases} \frac{dy}{dx} = y \\ \\ y(0) = 1 \end{cases}$$

$$\begin{aligned}
\int \frac{1}{y}\,dy &= \int dx \\
\ln|y| &= (x + c_1) \\
|y| &= e^{x+c_1} \\
&= e^{c_1}e^x \\
y &= \pm e^{c_1}e^x \\
&= Ce^x \\
y &= Ce^x \\
but\ y(0) &= 1 \\
1 &= Ce^0 \\
C &= 1 \\
y &= e^x
\end{aligned}$$

Now let's do it again using the Laplace transform

$$
\begin{aligned}
y'(x) &= y(x) \\
\mathcal{L}\{y'(x)\} &= \mathcal{L}\{y(x)\} \\
s\,\mathcal{L}\{y(x)\} - y(0) &= \mathcal{L}\{y(x)\} \\
s\,\mathcal{L}\{y(x)\} - \mathcal{L}\{y(x)\} &= y(0) \\
\mathcal{L}\{y(x)\}(s-1) &= 1 \\
\mathcal{L}\{y(x)\} &= \frac{1}{s-1} \\
\mathcal{L}^{-1}\{\mathcal{L}\{y(x)\}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\},\ a = 1 \\
y(x) &= e^{x}
\end{aligned}
$$

Example 3.13

$$
\textit{Solve}\ \begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 2,\ y'(0) = 0 \end{cases}
$$

$$
\begin{aligned}
\mathcal{L}\{y''\} + 2\,\mathcal{L}\{y'\} + 5\,\mathcal{L}\{y\} &= \mathcal{L}\{0\} \\
s^2\,\mathcal{L}\{y\} - sy(0) - y'(0) + 2\left(s\,\mathcal{L}\{y\} - y(0)\right)_5 \mathcal{L}\{y\} &= 0 \\
\mathcal{L}\{y\}\left(s^2 + 2s + 5\right) &= 2s + 4 \\
\mathcal{L}\{y\} &= \frac{2s+4}{s^2+2s+5} \\
\mathcal{L}^{-1}\left\{\mathcal{L}\{y\}\right\} &= \mathcal{L}^{-1}\left\{\frac{2s+4}{s^2+2s+5}\right\} \\
y(x) &= 2e^{-x}\cos(2x) + e^{-x}\sin(2x)
\end{aligned}
$$

Example 3.14

$$
\textit{Solve: } y'' + y = \sin(2t),\ y(0) = 2,\ y'(0) = 1
$$

$$
\begin{aligned}
\mathcal{L}\{y''\} - \mathcal{L}\{y\} &= \mathcal{L}\{\sin(2t)\} \\
s^2\,\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} &= \frac{2}{s^2+4},\ \ s > 0 \\
s^2\,\mathcal{L}\{y\} - 2s - 1 + \mathcal{L}\{y\} &= \frac{2}{s^2+4} \\
\mathcal{L}\{y\}(s^2+1) - 2s - 1 &= \frac{1}{s^2+4} \\
\mathcal{L}\{y\}(s^2+1) &= \frac{1}{s^2+4} + 2s + 1 \\
&= \frac{2 + 2s^3 + 8s + s^2 + 4}{s^2+4} \\
\mathcal{L}\{y\} &= \frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)} \\
y &= \mathcal{L}^{-1}\left\{\frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)}\right\} \\
\frac{2s^3 + s^2 + 8s + 6}{(s^2+4)(s^2+1)} &= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} \\
2s^3 + s^2 + 8s + 6 &= (as+b)(s^2+4) + (cs+d)(s^2+1) \\
&= as^3 + bs^2 + 4as + 4b + cs^3 + ds^2 + cs + d \\
&= s^3(a+c) + s^2(b+d) + s(4a+c) + (4b+d) \\
a + c &= 2 \\
b + d &= 1 \\
4a + c &= 8 \\
4b + d &= 6
\end{aligned}
$$

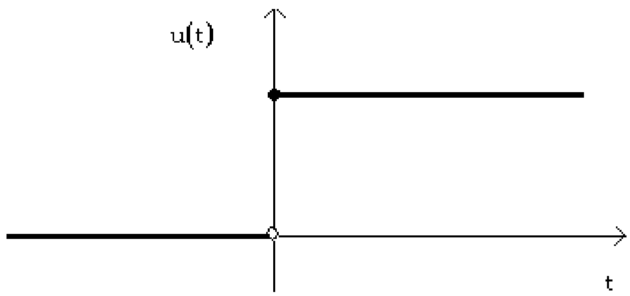
$$
\left[\begin{array}{cccc|c}
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 8 \\
0 & 4 & 0 & 1 & 6
\end{array}\right]
\sim
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 5/3 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2/3
\end{array}\right]
$$

$$
2s^3 + s^2 + 8s + 6 = 2s^3 + s^2(5/3 - 2/3) + 8s + 20/3 - 2/3
$$

3.3 Unit Step-function

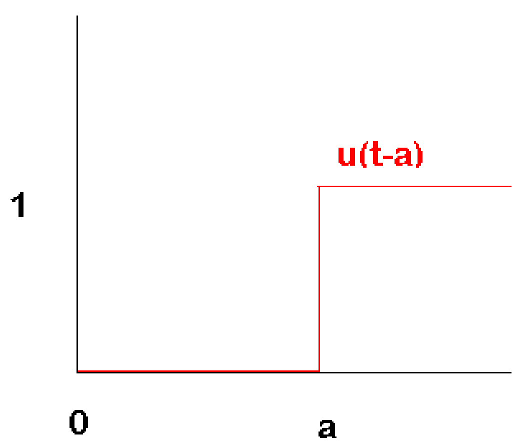
Definition The function $u(t)$ is called the unit step-function.

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{3.3}$$



$u_c(t)$ is a transformation of the unit step-function

$$u_c(t) = u(t-c) = \begin{cases} 0, & t - c < 0 \\ 1, & t - c > 0 \end{cases} \tag{3.4}$$

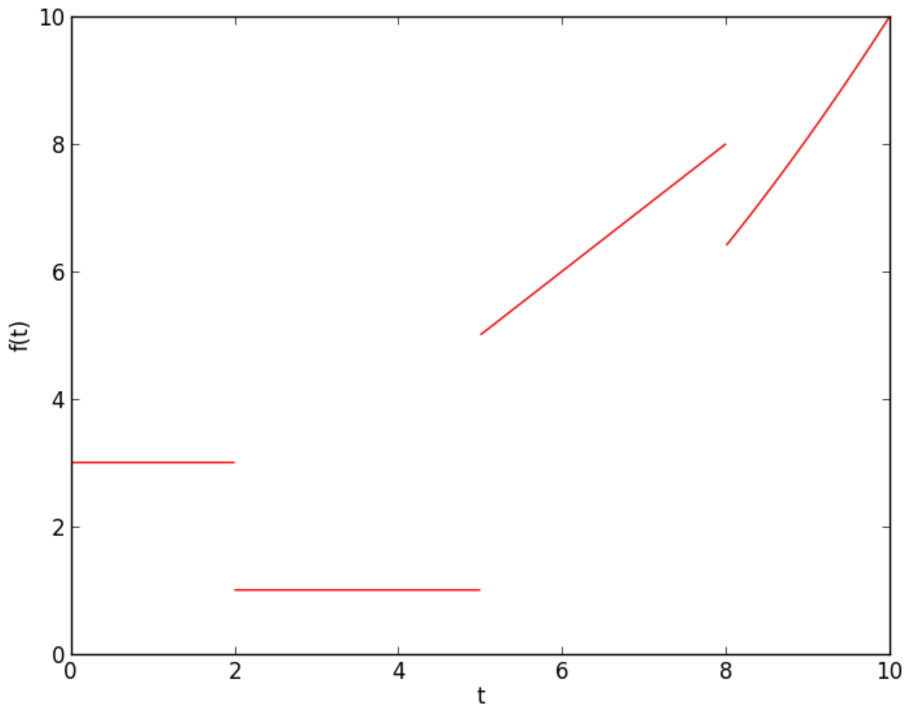


Example 3.15 *For $a < b$, find $u(t-a) - u(t-b)$.*

$$u(t-a) - u(t-b) = \begin{cases} 0, & (t < a) \cup (t > b) \\ 1, & a < t < b \end{cases}$$

Example 3.16

$$Draw\ f(t) = \begin{cases} 3, & t < 2 \\ 1, & 2 < t < 5 \\ t, & 5 < t < 8 \\ t^2/10, & t > 8 \end{cases}$$



Theorem 3.17 *For $C \geq 0$*

$$\mathcal{L}\{u_c(t)\} = \mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}, \; s > 0$$

Proof

$$
\begin{aligned}
\mathcal{L}\left\{u_c(t)\right\} &= \int_0^\infty u_c(t)e^{-st}\,dt \\
\text{Recall } u_c(t) &= \begin{cases} 0, & t < c \\ 1, & t > c \end{cases} \\
&= \int_0^c \cancelto{0}{u_c(t)} e^{-st}\,dt + \int_c^\infty \cancelto{1}{u_c(t)} e^{-st}\,dt \\
&= 0 + \int_c^\infty e^{-st}\,dt \\
&= \left[-\frac{e^{-st}}{s}\right]_c^\infty \\
c &= -\frac{1}{s}\left[0 - e^{-sc}\right] \\
&= \frac{e^{-sc}}{s},\ s > 0
\end{aligned}
$$

Example 3.18

$$
\begin{aligned}
\mathcal{L}\left\{u_c(t)f(t-c)\right\} &= e^{-cs}F(s),\ s > 0 \\
&= \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt \\
&= \int_0^c \cancelto{0}{u_c(t)} f(t-c)e^{-st}\,dt + \int_c^\infty \cancelto{1}{u_c(t)} f(t-c)e^{-st}\,dt \\
&= \int_c^\infty f(t-c)e^{-st}\,dt \\
\textit{Let } v = t - c & \\
&= \int_0^\infty f(v)e^{-s(v+c)}\,dv \\
&= \int_0^\infty f(v)e^{-sv}e^{-sc}\,dv \\
&= e^{-sc}\int_0^\infty f(v)e^{-sv}\,dv \\
&= e^{-sc}\,\mathcal{L}\left\{f(v)\right\} \\
&= e^{-sc}F(s)
\end{aligned}
$$

Example 3.19 *For homework:*

$$\mathcal{L}\left\{t^2u(t-1)\right\} = \mathcal{L}\left\{t^2u_1(t)\right\}$$

MISSING NOTES HERE

Example 3.20

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\int_0^t e^{-(t-\tau)}\sin\tau\,d\tau\right\} \\ &= \mathcal{L}\left\{e^{-t} * \sin t\right\} \\ &= \mathcal{L}\left\{e^{-t}\right\}\cdot\mathcal{L}\left\{\sin t\right\} \\ &= \left(\frac{1}{s+1}\right)\left(\frac{1}{s^2+1}\right) \\ &= \frac{1}{(s+1)(s^2+1)},\ \ s>0 \end{aligned}$$

Example 3.21

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\int_0^t \sin(t-\tau)\cos\tau\,d\tau\right\} \\ &= \mathcal{L}\left\{\sin t * \cos t\right\} \\ &= \left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right) \\ &= \frac{s}{(s^2+1)^2},\ \ s>0 \end{aligned}$$

Example 3.22

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} \\
\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} &= t \\
\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} &= \sin t \\
\mathcal{L}\left\{t\right\} &= \frac{1}{s^2} \\
\mathcal{L}\left\{\sin t\right\} &= \frac{1}{s^2+1} \\
\mathcal{L}\left\{t\right\}\mathcal{L}\left\{\sin t\right\} &= \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+1}\right) \\
\mathcal{L}\left\{t * \sin t\right\} &= \frac{1}{s^2(s^2+1)} \\
t * \sin t &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} \\
&= \int_0^t (t-\tau)\sin\tau\, d\tau \\
&\begin{cases} u = t-\tau, & u' = -d\tau \\ v = -\cos\tau, & v' = \sin\tau \end{cases} \\
&= \left[-\cos\tau(t-\tau)\right]_0^t - \int_0^t \cos\tau\, d\tau \\
&= t - \left[\sin\tau\right]_0^t = t - \sin t
\end{aligned}$$

Example 3.23

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+1}\right\} \\
F(s) &= \frac{1}{s^2+1} \\
f(t) &= \sin t \\
\sin t * g(t) &= \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+1}\right\} \\
&= \int_0^t \sin(t-\tau)g(\tau)\, d\tau
\end{aligned}$$

Example 3.24

$$\begin{aligned}
y'' + 4y &= g(t);\ \ y(0) = 3,\ \ y'(0) = -1 \\
\mathcal{L}\left\{y''\right\} + \mathcal{L}\left\{4y\right\} &= \mathcal{L}\left\{g(t)\right\} \\
s^2\,\mathcal{L}\left\{y\right\} - 3s + 1 + 4\,\mathcal{L}\left\{y\right\} &= G(s) \\
\mathcal{L}\left\{y\right\}(s^2+4) - 3s + 1 &= G(s) \\
\mathcal{L}\left\{y\right\} &= \frac{G(s) + 3s - 1}{s^2+4} \\
Let\ F(s) &= \frac{1}{s^2+4} \\
f(t) &= \frac{1}{2}\sin(2t) \\
\mathcal{L}^{-1}\left\{G(s)F(s)\right\} &= \frac{1}{2}\sin(2t) * g(t) \\
y &= \frac{1}{2}\sin(2t) * g(t) + \mathcal{L}^{-1}\left\{\frac{3s}{s^2+4} - \frac{1}{s^2+4}\right\} \\
&= \frac{1}{2}\sin(2t) * g(t) + 3\cos(2t) - \frac{1}{2}\sin(2t) \\
&= \frac{1}{2}\int_0^t g(t-\tau)\sin(2\tau)\,d\tau + 3\cos(2t) - \frac{1}{2}\sin(2t)
\end{aligned}$$

3.4 Gamma Function

$$\Gamma(a+1) = \int_0^\infty t^a e^{-t}\,dt,\ \ a > -1 \tag{3.5}$$

Theorem 3.25

$$\Gamma(a+1) = a\Gamma(a)$$

Example 3.26

$$\begin{aligned}\Gamma(4) &= 3\Gamma(3)\\ &= 3\cdot 2\Gamma(2)\\ &= 3\cdot 2\cdot 1\Gamma(1)\\ &= 3\cdot 2\cdot 1\cdot 1 = 6 = 3!\end{aligned}$$

So for positive integers, $\Gamma(a+1) = a!$

Proof

$$\begin{aligned}\Gamma(a+1) &= \int_0^\infty t^a e^{-t}\,dt\\ &\begin{cases} u = t^a, & u' = a\,t^{a-1}\\ v = -e^{-t}, & v' = e^{-t}\end{cases}\\ &= \cancelto{0}{\left[-t^a e^{-t}\right]_0^\infty} + a\int_0^\infty t^{a-1}e^{-t}\,dt\\ &= 0 + a\Gamma(a)\end{aligned}$$

Example 3.27

$$
\begin{aligned}
\Gamma(1) &= \Gamma(0+1) \Rightarrow a = 0 \\
&= \int_0^\infty t^0 e^{-t}\,dt \\
&= \int_0^\infty e^{-t}\,dt \\
&= \left[-e^{-t}\right]_0^\infty \\
&= 1 \\
\Gamma(3) &= \Gamma(2+1) \\
&= 2\Gamma(2) \\
&= 2 \cdot 1 = 2 \\
\Gamma(4) &= \Gamma(3+1) \\
&= 3\Gamma(2+1) \\
&= 3 \cdot 2\Gamma(1+1) \\
&= 3! \\
\Gamma(n+1) &= n!,\ \ n \in \mathbb{Z}^+
\end{aligned}
$$

Proof

$$\Gamma\left(\frac{1}{2}\right) = \Gamma\left(-\frac{1}{2}+1\right)$$

$$= \int_0^\infty t^{-1/2} e^{-t}\,dt$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(\int_0^\infty t^{-1/2} e^{-t}\,dt\right)^2$$

Let $u^2 = t,\ 2u\,du = dt$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left(\int_0^\infty u^{-1} e^{-u^2} \cdot 2u\,du\right)^2$$

$$= \left(2\int_0^\infty e^{-u^2}\,du\right)^2$$

$$= \left(\int_{-\infty}^\infty e^{-u^2}\,du\right)^2$$

$$= \left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\,dx\,dy$$

In cylindrical coordinates: $x^2 + y^2 = r^2$

$$= \int_0^{2\pi} \int_0^\infty e^{-r^2} r\,dr\,d\theta$$

(Let $w = r^2,\ dw = 2r\,dr$)

$$= \frac{1}{2}\int_0^{2\pi} \int_0^\infty e^{-w}\,dw\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1\,d\theta$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{1}{2}(2\pi) = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 3.28

$$\begin{aligned}
\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2}+1\right) \\
&= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{2} \\
\Gamma\left(\frac{7}{2}\right) &= \Gamma\left(\frac{5}{2}+1\right) \\
&= \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right) \\
&= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) \\
&= \frac{15}{4}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{15\sqrt{\pi}}{8}
\end{aligned}$$

Example 3.29

$$\begin{aligned}
\Gamma\left(-\frac{5}{3}\right) &= \Gamma\left(-\frac{8}{3}+1\right) \\
&= -\frac{8}{3}\Gamma\left(\frac{-11}{3}+1\right) \\
&= \frac{88}{9}\Gamma\left(\frac{-14}{3}+1\right)
\end{aligned}$$

Notice that it's just getting more and more complicated. So we need to do the opposite of what we've *been doing*

$$
\begin{aligned}
-\frac{5}{3}\,\Gamma\left(-\frac{5}{3}\right) &= \Gamma\left(-\frac{5}{3}+1\right) = \Gamma\left(-\frac{2}{3}\right)\\
\Gamma\left(-\frac{5}{3}\right) &= -\frac{3}{5}\,\Gamma\left(-\frac{2}{3}\right)\\
-\frac{2}{3}\,\Gamma\left(-\frac{2}{3}\right) &= \Gamma\left(-\frac{2}{3}+1\right) = \Gamma\left(\frac{1}{3}\right)\\
\Gamma\left(-\frac{2}{3}\right) &= -\frac{3}{2}\,\Gamma\left(\frac{1}{3}\right)\\
\Gamma\left(-\frac{5}{3}\right) &= \frac{9}{10}\,\Gamma\left(\frac{1}{3}\right)
\end{aligned}
$$

Example 3.30

$$
\begin{aligned}
\Gamma\left(-\frac{13}{7}\right) &= -\frac{7}{13}\Gamma\left(-\frac{6}{7}\right)\\
&= \frac{49}{78}\,\Gamma\left(\frac{1}{7}\right)
\end{aligned}
$$

4 Systems of Differential Equations

Example 4.1

$$
\begin{cases}
\frac{dx}{dt} = 3x + 5y\\
\frac{dy}{dt} = 3x + y\\
x(0) = 1,\ y(0) = 2
\end{cases}
$$

$$\begin{cases} \mathcal{L}\left\{x'(t)\right\} = 3\,\mathcal{L}\left\{x\right\} + 5\,\mathcal{L}\left\{y\right\} \\ \mathcal{L}\left\{y'(t)\right\} = 3\,\mathcal{L}\left\{x\right\} + \mathcal{L}\left\{y\right\} \end{cases}$$

$$\begin{cases} s\,\mathcal{L}\left\{x\right\} - \cancelto{1}{x(0)} = 3\,\mathcal{L}\left\{x\right\} + 5\,\mathcal{L}\left\{y\right\} \\ s\,\mathcal{L}\left\{y\right\} - \cancelto{2}{y(0)} = 3\,\mathcal{L}\left\{x\right\} + \mathcal{L}\left\{y\right\} \end{cases}$$

$$\begin{cases} (s-3)\,\mathcal{L}\left\{x\right\} - 5\,\mathcal{L}\left\{y\right\} = 1 \\ -3\,\mathcal{L}\left\{x\right\} + (s-1)\,\mathcal{L}\left\{y\right\} = 2 \end{cases}$$

$$\begin{bmatrix} (s-3) & -5 \\ -3 & (s-1) \end{bmatrix} \begin{bmatrix} \mathcal{L}\left\{x\right\} \\ \mathcal{L}\left\{y\right\} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \mathcal{L}\left\{x\right\} \\ \mathcal{L}\left\{y\right\} \end{bmatrix} &= \begin{bmatrix} (s-3) & -5 \\ -3 & (s-1) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{s^2-4s-12} \begin{bmatrix} (s-1) & 5 \\ 3 & (s-3) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \frac{1}{s^2-4s-12} \begin{bmatrix} (s+9) \\ (2s-3) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left\{x\right\} &= \frac{s+9}{s^2-4s-12} \\ \mathcal{L}\left\{y\right\} &= \frac{2s-3}{s^2-4s-12} \\ x &= \frac{1}{8}e^{-2t}\left(15e^{8t}-7\right) \\ y &= \frac{1}{8}e^{-2t}\left(9e^{8t}+7\right) \end{aligned}$$

Example 4.2

$$\begin{cases} x'(t) = 3x - 5y \\ y'(t) = x + 5y \\ x(0) = -3,\ y(0) = 4 \end{cases}$$

$$\begin{cases} \mathcal{L}\{x'(t)\} = 3\,\mathcal{L}\{x\} - 5\,\mathcal{L}\{y\} \\ \mathcal{L}\{y'(t)\} = \mathcal{L}\{x\} + 5\,\mathcal{L}\{y\} \end{cases}$$

$$\begin{cases} s\,\mathcal{L}\{x\} - \cancelto{-3}{x(0)} = 3\,\mathcal{L}\{x\} - 5\,\mathcal{L}\{y\} \\ s\,\mathcal{L}\{y\} - \cancelto{4}{y(0)} = \mathcal{L}\{x\} + 5\,\mathcal{L}\{y\} \end{cases}$$

$$\begin{cases} (s-3)\,\mathcal{L}\{x\} + 5\,\mathcal{L}\{y\} = -3 \\ -\,\mathcal{L}\{x\} + (s-5)\,\mathcal{L}\{y\} = 4 \end{cases}$$

$$\begin{bmatrix} (s-3) & 5 \\ -1 & (s-5) \end{bmatrix} \begin{bmatrix} \mathcal{L}\{x\} \\ \mathcal{L}\{y\} \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{L}\{x\} \\ \mathcal{L}\{y\} \end{bmatrix} = \frac{1}{s^2-8s+20} \begin{bmatrix} (s-5) & -5 \\ 1 & (s-3) \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{s^2-8s+20} \begin{bmatrix} -3s-5 \\ 4s-15 \end{bmatrix}$$

$$\mathcal{L}\{x\} = \frac{-3s-5}{s^2-8s+20}$$

$$\mathcal{L}\{y\} = \frac{4s-15}{s^2-8s+20}$$

$$x = -\frac{1}{2}e^{4t}(17\sin(2t) + 6\cos(2t))$$

$$y = \frac{1}{2}e^{4t}(\sin(2t) + 8\cos(2t))$$

Example 4.3

$$\int e^{x^2}\,dx$$

We can't integrate this using the techniques we learned in Calculus, but we can approximate it as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n,\ \ a_n = \frac{f^{(n)}(a)}{n!}$$

So we can write $f(x)$ as

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots + a_n(x-a)^n$$

Example 4.4

$$y'' - xy = 0,\ \ y(0) = 1,\ \ y'(0) = 2$$

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots \end{aligned}$$

Now, we know $y(0)$ and $y'(0)$, but nothing higher, so first, let's plug in our initial conditions to the original equation to find $y''(0)$

$$\begin{aligned} y''(x) - xy(x) &= 0 \\ y''(0) - 0 \cdot \cancelto{1}{y(0)} &= 0 \\ y''(0) &= 0 \end{aligned}$$

Now let's take the derivative of the equation, and plug in our initial conditions to find $y'''(0)$

$$\begin{aligned} y'''(x) - y(x) - xy'(x) &= 0 \\ y'''(0) - \cancelto{1}{y(0)} - 0 \cdot \cancelto{2}{y'(0)} &= 0 \\ y'''(0) &= 1 \end{aligned}$$

Now let's do it for the fourth derivative

$$\begin{aligned} y^{(4)}(x) - y'(x) - y'(x) - xy''(x) &= 0 \\ y^{(4)}(x) - 2y'(x) - xy''(x) &= 0 \\ y^{(4)}(0) - 2\cancelto{2}{y'(0)} - 0 \cdot \cancelto{1}{y''(0)} &= 0 \\ y^{(4)}(0) &= 4 \end{aligned}$$

We can keep going, but we'll stop here with our approximation, giving us

$$y(x) = 1 + 2x + \frac{1}{6}x^3 + \frac{4}{24}x^4 + \ldots$$

Example 4.5

$$y'' - x^2x = 0,\ y(0) = 1,\ y'(0) = 0$$

$$\begin{aligned}
y'(x) &= \sum_{n=1}^{\infty} n\, a_n x^{n-1} \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
y''(x) - x^2 y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n = 0 \\
&= (2)(1)a_2 + (3)(2)a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} a_{n-2}x^n = 0 \\
&= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} x^n[(n+2)(n+1)a_{n+2} - a_{n-2}] = 0
\end{aligned}$$

$$2a_2 = 0,\ 6a_3 = 0,\ (n+2)(n+1)a_{n+2} - a_{n-2} = 0,\ n \geq 2$$

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)},\ n \geq 2$$

Example 4.6

$$\textit{Solve } (-4x^2 - 1)y'' + 5xy' = 0;\ y(0) = -3,\ y'(0) = -8$$

$$\textit{Recall } \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \sum_{n=0}^{\infty} a_n(x-a)^n$$

We already know $y(0)$ *and* $y'(0)$, *so it would be simplest to let* $a = 0$, *as we need to know* $y(a)$ *and* $y'(a)$. *We're also going to need further derivatives, so lets find* $y''(0)$ *by evaluating the equation with* $x = 0$.

$$(-1)y''(0) + 0 = 0 \Rightarrow y''(0) = 0$$

Now we need the third derivative

$$-8xy'' + (-4x^2 - 1)y''' + 5y' + 5xy'' = 0$$

and evaluate at $x = 0$

$$0 + (-1)y'''(0) + 5(-8) + 0 = 0 \Rightarrow y'''(0) = -40$$

Now let's use the power series

$$\begin{aligned} y(x) &= \frac{y(0)}{0!} + \frac{y'(0)}{1!}x' + \frac{y''(0)}{2!}x^2 + \dots \\ &= -3 - 8x - \frac{20}{3}x^3 + \dots \end{aligned}$$

Example 4.7

$$\textit{Solve } (-3x + 6)y'' + 8y' + (2x - 5)y = 0; \; y(0) = 5, \; y'(0) = 0$$

First, evaluate at $x = 0$ to find $y''(0)$

$$6y''(0) + 0 - 5\cancelto{5}{y'(0)} + (-5)\cancelto{0}{y(0)} = 0 \Rightarrow y''(0) = \frac{25}{6}$$

Now take the derivative and evaluate at $x = 0$ to find $y'''(0)$

$$\begin{aligned} -3y''(x) + (-3x + 6)y'''(x) + 8y''(x) + 2y(x) + (2x - 5)y'(x) &= 0 \\ -3\cancelto{25/6}{y''(0)} + 6y'''(0) + 8\cancelto{25/6}{y''(0)} + 2\cancelto{5}{y(x)} - 5\cancelto{0}{y'(x)} &= 0 \\ -\frac{75}{6} + 6y'''(0) + \frac{100}{3} + 10 &= 0 \\ y'''(0) &= -\frac{185}{36} \end{aligned}$$

We can go further, but if we stop here, we have

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

$$y(x) = 5 + \frac{25}{12}x^2 - \frac{185}{216}x^3 + \dots$$

Example 4.8

$$\textit{Solve } xy'' + y' + y = 0;\ y(0) = -1,\ y'(0) = 1$$

Radius of Convergence

Definition Let $p(x)$, $q(x)$, and $r(x)$ be analytic functions. A point x_0 is a singular point for the differential equation

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = 0 \tag{4.1}$$

if $\dfrac{q(x_0)}{p(x_0)}$ or $\dfrac{r(x_0)}{p(x_0)}$ is undefined. Otherwise, x_0 is called a *regular point.*

Try solving the following:

$$\begin{cases} xy'' + y' + y = 0 \\ y(0) = -1,\ y'(0) = 1 \end{cases}$$

If we try evaluating at $x = 0$ to find $y''(0)$, we end up multiplying the $y''(0)$ by zero, so we can't get it by that method. We could rewrite this problem by dividing everyting by x

$$y'' + \frac{1}{x}y' + \frac{1}{x}y = 0$$

Now we see that $x = 0$ is a singular point.

Find the singular points for the following equations:

$$x^2y'' + xy' = 0 \Rightarrow x_0 = 0$$

$$2y'' + \frac{y}{x-1} = 0 \Rightarrow x_0 = 1$$

Theorem 4.9 *Suppose*

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is a series solution do Equation 4.1. Then, the radius of convergence for $y(x)$ is at least as large as the distance from x_0 (regular point) to the nearest singular point of the equation.

Example 4.10 *Find the radius of convergence about x_0*

$$(x+1)y'' + y' + (x+1)y = 0, \ \textit{about } x_0 = 0$$

So first we divide by $(x+1)$

$$y'' + \frac{y'}{x+1} + y = 0$$

$x = -1$ is a singular point. So the radius of convergence is at least 1.

Example 4.11 *Find the radius of convergence about x_0*

$$(x^2 - 2x - 3)y'' + (x+1)y' + 3y = 0, \ \textit{about } x_0 = 0$$

First divide by $(x+1)$, to get singular point $x = -1$. Then divide the original equation by (x^2-2x-3) to get

$$y'' + \frac{\cancel{(x+1)}y'}{(x-3)\cancel{(x+1)}} + \frac{3y}{(x-3)(x+1)} = 0$$

which has singular points $x = -1$ and $x = 3$. So the radius of convergence is at least 1.

Example 4.12

$$(x^3 + x^2 + x + 1)y'' + 2xy' + (x-1)y = 0, \ \textit{about } x_0 = 0$$

Factor the big polynomial first

$$x^3 + x^2 + x + 1 = x^2(x+1) + 1(x+1) = (x^2+1)(x+1)$$

Now divide that out to get

$$y'' + \frac{2xy'}{(x^2+1)(x+1)} + \frac{(x-1)y}{(x^2+1)(x+1)} = 0$$

$x = -1$, $x = \pm\,\iota$. *Radius of convergence is at least* $\sqrt{2}$

5 Final Examples

Example 5.1

$$\mathcal{L}\left\{t\cosh(t)\right\}$$

$$\begin{aligned}
\cosh(t) &= \frac{e^t+e^{-t}}{2} \\
F(s) &= \frac{1}{2}\,\mathcal{L}\left\{te^t+te^{-t}\right\} \\
&= \frac{1}{2}\,\mathcal{L}\left\{te^t\right\}+\frac{1}{2}\,\mathcal{L}\left\{te^{-t}\right\} \\
\mathcal{L}\left\{t^n e^{at}\right\},\ n\in\mathbb{Z}^+ &= \frac{n!}{(s-a)^{n+1}},\ \ s>a \\
F(s) &= \frac{1}{2}\frac{1!}{(s-1)^{1+1}}+\frac{1}{2}\frac{1!}{(s+1)^{1+1}},\ \ s>1 \\
&= \frac{1}{2}\left(\frac{1}{(s-1)^2}+\frac{1}{(s+1)^2}\right),\ \ s>1
\end{aligned}$$

Example 5.2

$$\mathcal{L}\left\{f(t)\right\},\ \ \textit{where}\ f(t)=3\cos(2t)+\int_0^t e^{2(t-\tau)}\tau\sin(3\tau)\,d\tau$$

$$\begin{aligned}
\mathcal{L}\left\{f(t)\right\} &= 3\,\mathcal{L}\left\{\cos(2t)\right\}+\mathcal{L}\left\{\int_0^t e^{2(t-\tau)}\tau\sin(3\tau)\,d\tau\right\} \\
&= 3\frac{s}{s^2+4}+\mathcal{L}\left\{e^{2t}*t\sin(3t)\right\} \\
&= 3\frac{s}{s^2+4}+\mathcal{L}\left\{e^{2t}\right\}\mathcal{L}\left\{t\sin(3t)\right\} \\
&= 3\frac{s}{s^2+4}+\frac{1}{s-2}\frac{6s}{(s^2+9)^2} \\
&= 3\frac{s}{s^2+4}+\frac{6s}{(s-2)(s^2+9)^2}
\end{aligned}$$

Example 5.3

$$\mathcal{L}^{-1}\left\{\frac{2s+9}{s^2+8s+17}\right\}$$

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\left\{\frac{2s+9}{s^2+8s+16+1}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+9}{(s+4)^2+1^2}\right\} \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{s}{(s+4)^2+1^2}\right\} + 9\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+1^2}\right\} \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{s+4-4}{(s+4)^2+1^2}\right\} + 9\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+1^2}\right\} \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{s+4}{(s+4)^2+1^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2+1^2}\right\} \\
&= 2e^{-4t}\cos(t) + e^{-4t}\sin(t) = e^{-4t}\left(2\cos(t)+\sin(t)\right)
\end{aligned}$$

Example 5.4

$$y'' + 6y' + 8y = \delta(t),\ \ y(0)=0,\ \ y'(0)=0$$

$$\begin{aligned}
s^2\,\mathcal{L}\{y\} + 6s\,\mathcal{L}\{y\} + 8\,\mathcal{L}\{y\} &= e^{-0s} \\
\mathcal{L}\{y\}\,(s^2+6s+8) &= 1 \\
\mathcal{L}\{y\} &= \frac{1}{s^2+6s+8} \\
&= \frac{1}{(s+2)(s+4)} \\
\frac{1}{(s+2)(s+4)} &= \frac{A}{s+2} + \frac{B}{s+4} \\
1 &= A(s+4) + B(s+2) \\
A = {}^1\!/\!_2,\ B &= {}^{-1}\!/\!_2 \\
\mathcal{L}\{y\} &= \frac{1}{2}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s+4} \\
y &= \frac{1}{2}\left(e^{-2t} - e^{-4t}\right)
\end{aligned}$$

Example 5.5

$$y'' + 2xy' + 4y = 0,\ y(0) = 1,\ y'(0) = 2$$

$$\textit{Just use: } \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$

The solution is:

$$y(x) = 1 + 2x - 2x^2 - 2x^3 + \frac{4}{3}x^4 + \dots$$

Example 5.6 *Find the radius of convergence for the series solution about $x_0 = 0$ to the equation:*

$$(x^2 + 2x + 10)(x^2 + 6x + 8)y'' + 2xy' - 4y = 0$$

$$y'' + \frac{2xy'}{(x+2)(x+4)(x^2+2x+10)} - \frac{4y}{(x+2)(x+4)(x^2+2x+10)} = 0$$

You can actually just use the quadratic formula. The roots will be the singular points, so the singular points are $x = -1 \pm 3i$ and $x = -4, -2$. The closest point is $-1 - 3i$, and it is 2 away from 0, so the radius of convergence is at least 2.

Example 5.7

$$y'' + 3y' + 2y = \frac{1}{1 + e^x} = g$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -2,\ \lambda = -1$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x} = y_1 + y_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$y_p = -y_1 \int \frac{y_2 \cdot g}{W} + y_2 \int \frac{y_1 \cdot g}{W}$$