Partial Differential Equations Notes

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Spring 2014

Contents

0 Class Overview

0.1 Goals

Linear PDE, including

1. Laplace's Equation
2. Heat Equation
3. Wave Equation

3 Second Order Linear Equations

ODE	PDE
One indep. variable	Multiple indep. variables

$$P(t)y''(t) + Q(t)y'(t) + R(t)y = g(t) \tag{3.1}$$

If $g(t) = 0$, then Equation 3.1 is called *homogeneous*. Otherwise it is non-homogeneous or heterogeneous.

$$ay'' + by' + cy = 0 \tag{3.2}$$

We solve this by setting up the characteristic equation, and solving for r

$$ar^2 + br + c = 0 \tag{3.3}$$

1. Distinct roots $r = r_1,\ r = r_2$: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
2. Repeated roots $r = r_1 = r_2$: $y = c_1 e^{rt} + c_2 t e^{rt}$
3. Complex roots $r = \lambda \pm \mu\imath$: $y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$

10 Partial Differential Equations and Fourier Series

10.1 Two-Point Boundary Value Problems

Recall initial value problems from earlier, which took the form

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(0) = y_0, \; y'(0) = y_1 \end{cases}$$

Now we will be working with slightly different problems, which take the form

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x) \\ y(\alpha) = y_0, \; y(\beta) = y_1 \end{cases}$$

Boundary value problems may have

- unique solutions
- infinitely many solutions
- no solutions

This boundary value problem (BVP) is homogeneous only when $g(x) = 0$, $y_0 = 0$, and $y_1 = 0$. Otherwise, it is a non-homogeneous BVP.

Example 10.1.

$$\begin{cases} y'' + 2y = 0 \\ y(0) = 1; \; y(\pi) = 0 \end{cases}$$

$$r^2 + 2 = 0 \implies r = \underbrace{0}_{\lambda} \pm \underbrace{\sqrt{2}}_{\mu}\imath$$

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

$$y(0) = 1 = c_1 + 0 \implies c_1 = 1$$

$$y(\pi) = 0 = \cos(\sqrt{2}\pi) + c_2 \sin(\sqrt{2}\pi)$$

$$c_2 = -\frac{\cos(\sqrt{2}\pi)}{\sin(\sqrt{2}\pi)} = -\cot(\sqrt{2}\pi)$$

$$y(x) = \cos(\sqrt{2}x) - \cot(\sqrt{2}\pi)\sin(\sqrt{2}x)$$

Example 10.2.

$$\begin{cases} y'' + y = 0 \\ y(0) = 1;\ y(\pi) = a \end{cases}$$

$$r^2 + 1 = 0 \implies r = \pm\imath$$

$$y = c_1 \cos(x) + c_2 \sin(x)$$

$$y(0) = 1 = c_1 + 0 \implies c_1 = 1$$

$$y(\pi) = a = \cos(\pi) + c_2 \sin(\pi) = -1 + 0 \implies a = -1$$

$$\begin{cases} y = \cos(x) + c_2 \sin(x) & \text{if } a = -1 \\ \text{no solution} & \text{if } a \neq -1 \end{cases}$$

Example 10.3.

$$\begin{cases} y'' + 4y = \sin(x) \\ y(0) = 0;\ y(\pi) = 0 \end{cases}$$

$$r^2 + 4 = 0 \implies r = \pm 2\imath$$

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

$$y_p = A\sin(x) + B\cos(x)$$

$$y_p' = A\cos(x) - B\sin(x)$$

$$y_p'' = -A\sin(x) - B\cos(x)$$

$$-A\sin(x) - B\cos(x) + 4A\sin(x) + 4B\cos(x) = 3A\sin(x) + 3B\cos(x) = \sin(x)$$

$$3A = 1 \implies A = \frac{1}{3},\ 3B = 0 \implies B = 0$$

$$y_p = \frac{1}{3}\sin(x)$$

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\sin(x)$$

$$y(0) = 0 = c_1$$

$$y(\pi) = 0 = 0 + c_2\sin(2\pi) + \frac{1}{3}\sin(\pi) = 0$$

$$y(x) = c_2\sin(2x) + \frac{1}{3}\sin(x)$$

Eigenvalue Problem

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

$\boldsymbol{x} = 0$ is a trivial solution. λ is called an eigenvalue for non-trivial solutions. $\boldsymbol{x}$ is an eigenvector. $c\boldsymbol{x}$ is also an eigenvector.

Example 10.4. Find the eigenvalue and eigenvectors for

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0;\ y(\pi) = 0 \end{cases}$$

case 1: $\lambda > 0$

$$r^2 + \lambda = 0 \implies r = \pm\sqrt{\lambda}\imath$$

$$y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$y(0) = 0 = c_1 + 0$$

$$y = c_2\sin(\sqrt{\lambda}x)$$

$$y(\pi) = 0 = c_2\sin(\sqrt{\lambda}\pi)$$

$$\text{for a non-trivial solution, we want } \sin(\sqrt{\lambda}\pi) = 0$$

$$\sqrt{\lambda}\pi = n\pi,\ n \in \mathbb{Z}$$

$$\sqrt{\lambda} = n \implies \lambda_n = n^2,\ n \in \mathbb{Z},\ n > 0$$

$$\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = 9,\ \lambda_n = n^2$$

$$y = c_2 \sin(\sqrt{\lambda}x) = c_2 \sin(nx)$$

$$\sin(nx) \text{ is an eigenvector}$$

case 2: $\lambda < 0$

$$r^2 + \lambda = 0 \implies r^2 = -\lambda,\ \text{let } \mu = -\lambda$$

$$r = \pm\sqrt{\mu}$$

$$y(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

$$y(0) = 0 = c_1 + c_2 \implies c_1 = -c_2$$

$$y(\pi) = 0 = c_1 e^{\sqrt{\mu}\pi} + c_2 e^{-\sqrt{\mu}\pi}$$

$$0 = -c_2 e^{\sqrt{\mu}\pi} + c_2 e^{-\sqrt{\mu}\pi} = -c_2 e^{\sqrt{\mu}\pi} + \frac{c_2}{e^{\sqrt{\mu}\pi}}$$

$$0 = \frac{-c_2 e^{2\sqrt{\mu}\pi} + c_2}{e^{\sqrt{\mu}\pi}} = -c_2 e^{2\sqrt{\mu}\pi} + c_2 = c_2\left(1 - e^{2\sqrt{\mu}\pi}\right)$$

$$1 - e^{2\sqrt{\mu}\pi} = 0 \implies e^{2\sqrt{\mu}\pi} = 1 \implies \mu = 0$$

but we already said $\mu = -\lambda$ and $\lambda < 0$, so instead c_2 must be zero

$$c_2 = -c_1 \implies c_1 = 0 \implies y(x) = 0 \text{ (trivial solution)}$$

So there are no eigenvectors for case 2.

case 3: $\lambda = 0$

$$y'' = 0 \implies y' = k_1 \implies y = k_1 x + k_2$$

$$y(0) = 0 = k_2,\ y(\pi) = 0 = k_1\pi \implies k_1 = 0$$

$$y(x) = 0 \text{ (trivial solution)}$$

Example 10.5.

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0;\ y'(\pi) = 0 \end{cases}$$

case 1: $\lambda > 0$

$$r^2 + \lambda = 0 \implies r = \pm\sqrt{\lambda}\imath$$

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$y(0) = 0 = c_1$$

$$y'(\pi) = 0 = 0 + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)$$

$$\text{let } \cos(\sqrt{\lambda}\pi) = 0$$

$$\cos(\theta) = 0 \text{ for } \theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\}$$

$$\text{so } \theta = \frac{2n-1}{2}\pi = \sqrt{\lambda_n}\pi \implies \lambda_n = \left(\frac{2n-1}{2}\right)^2$$

$$y_n = \sin\left(\frac{2n-1}{2}x\right) \text{ for } n = 1, 2, 3, \ldots$$

case 2: $\lambda < 0$ (incomplete, something seems off)

$$\mu := -\lambda;\ r = \pm\sqrt{\mu}$$

$$y(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x)$$

$$y(0) = 0 = c_1$$

$$y(x) = c_2 \sin(\sqrt{\mu}x) \implies y'(x) = c_2\sqrt{\mu}\cos(\sqrt{\mu}x)$$

$$y'(\pi) = 0 = c_2\sqrt{\mu}\cos(\sqrt{\mu}\pi)$$

if c_2 is non-zero, then the rest of the expression must be zero, so

$$\sqrt{\mu}\cos(\sqrt{\mu}\pi) = 0 \implies \cos(\sqrt{\mu}\pi) = 0$$

$$\cos(\pi x) = 0 \text{ for } x \text{ in } \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}$$

10.2 Fourier Series

Also called Trigonometric Series.

Definition 10.1 (periodic)**.** A function f is said to be *periodic* with period $T > 0$ if $f(x+T) = f(x)$

Remark. If T is a period of the function f, then $S \in \{nT \mid n \in \mathbb{Z}^+\}$ is also a period of the function f.

Definition 10.2 (fundamental period)**.** The smallest T that is a period of the function f is the fundamental period of f.

Remark. Let f and g have a period T.

1. Any *linear combination* of f and g has a period T.

2. fg also has a period T.

Proof.

1. f and g have period T means $f(x+T) = f(x)$ & $g(x+T) = g(x)$. We want to show that $af(x+T) + bg(x+T) = af(x) + bg(x)$

$$\begin{aligned} \text{Let } F(x) &= af(x) + bg(x) \\ &= af(x+T) + bg(x+T) \text{ because } f \text{ and } g \text{ are periodic with period } T. \\ &= F(x+T) \end{aligned}$$

2. f and g have period T means $f(x+T) = f(x)$ & $g(x+T) = g(x)$. We want to show that

$$af(x+T)\,bg(x+T) = af(x)\,bg(x)$$

$$\begin{aligned} \text{Let } F(x) &= af(x)\,bg(x) \\ &= af(x+T)\,bg(x+T) \text{ because } f \text{ and } g \text{ are periodic with period } T. \\ &= F(x+T) \end{aligned}$$

□

Definition 10.3 (inner product)**.** The standard inner product for two real-valued $u(x)$ & $v(x)$ for $\alpha \leq x \leq \beta$.

$$< u,\, v >= \int_{\alpha}^{\beta} u(x)v(x)\, \mathrm{d}\, x$$

Definition 10.4 (orthogonal)**.** $u \perp v$ if $< u,\, v >= 0$.

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}\, x = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases} \tag{10.1}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}\, x = 0\ \forall\ m,\ n \tag{10.2}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}\, x = \begin{cases} 0, & m = n \\ L, & m \neq n \end{cases} \tag{10.3}$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, \mathrm{d}\, x \tag{10.4}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, \mathrm{d}\, x \tag{10.5}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, \mathrm{d}\, x \tag{10.6}$$

Note that when $n = 0$, $a_n = a_0$.

Example 10.6. Find the period of $\cos\left(\frac{m\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$.

$$\frac{m\pi}{L}T = 2\pi \implies \frac{mT}{L} = 2$$
$$T = \frac{2L}{m} \text{ (is a period)} \implies mT = 2L \text{ (is a period)}$$

Example 10.7. Know that $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x = \int_{-L}\frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right)\right]\mathrm{d}\,x$$
$$= \frac{1}{2}\left[\sin\left(\frac{(m-n)\pi x}{L}\right)\frac{L}{(m-n)\pi} + \sin\left(\frac{(m+n)\pi x}{L}\right)\frac{L}{(m+n)\pi}\right]_{-L}^{L} = 0, \text{ if } m \neq n$$

If $m = n$, then the two cosine terms are the same, so it is $\cos^2$

$$\int_{-L}^{L}\cos^2\left(\frac{m\pi x}{L}\right) = \int_{-L}^{L}\frac{1+\cos\left(\frac{2m\pi x}{L}\right)}{2}\,\mathrm{d}\,x = \frac{1}{2}\left[x + \sin\left(\frac{2m\pi x}{L}\right)\frac{L}{2m\pi}\right]_{-L}^{L} = \frac{1}{2}[2L+0] = L$$

Definition 10.5 (Euler-Fourier Formula)**.**

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right]$$

a_0, a_m, and b_m are the fourier coefficients.

$$\int_{-L}^{L} f(x)\,\mathrm{d}\,x = \int_{-L}^{L}\frac{a_0}{2}\,\mathrm{d}\,x + \int_{-L}^{L}\sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right]\mathrm{d}\,x$$
$$\int_{-L}^{L} f(x)\,\mathrm{d}\,x = \frac{a_0}{2}\int_{-L}^{L}\mathrm{d}\,x + \sum_{m=1}^{\infty}\left[a_m\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}\,x + b_m\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}\,x\right]$$
$$\int_{-L}^{L} f(x)\,\mathrm{d}\,x = a_0L + 0 + 0 \implies a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}\,x$$
$$f(x)\cos\left(\frac{n\pi x}{L}\right) = \frac{a_0}{2}\cos\left(\frac{n\pi x}{L}\right) + \sum_{m=1}^{\infty} a_m\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_m\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

Now integrate both sides from $-L$ to L. Note that the cosine integral matches Equation 10.1, so that means it is either 0 if $m \neq n$ or L if $m = n$. Since n is a fixed, positive integer, $m = n$ only occurs once, so the whole series can be reduced to $a_n L$. Equation 10.2 shows that the last term is zero.

$$\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \,\mathrm{d}\,x = 0 + a_n L + 0$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \,\mathrm{d}\,x$$

Example 10.8. Find the fourier series for f:

$$f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x, & 0 \leq x < 2 \end{cases}$$
$$f(x+4) = f(x)$$

Note that $T = 4$ and $L = 2$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x) \,\mathrm{d}\,x = \frac{1}{2} \left[\int_{-2}^{0} (-x) \,\mathrm{d}\,x + \int_{0}^{2} x \,\mathrm{d}\,x \right] = 2$$
$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{n\pi x}{2}\right) \,\mathrm{d}\,x = \frac{1}{2} \left[\int_{-2}^{0} (-x) \cos\left(\frac{n\pi x}{2}\right) \,\mathrm{d}\,x + \int_{0}^{2} x \cos\left(\frac{n\pi x}{2}\right) \,\mathrm{d}\,x \right]$$
$$a_n = \frac{4}{(n\pi)^2} (\cos(n\pi) - 1) = \begin{cases} \dfrac{-8}{(n\pi)^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$b_n = 0$$

$$f(x) = \frac{2}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-8}{(n\pi)^2} \cos\left(\frac{n\pi x}{2}\right) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} (2n-1)^{-2} \cos\left(\frac{(2n-1)\pi x}{2}\right)$$

Example 10.9.

$$f(x+2) = f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 1-x, & 0 \leq x < 1 \end{cases}$$

$T = 2,\ L = 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\ \mathrm{d}\,x = \int_{-1}^{0} x+1\ \mathrm{d}\,x + \int_{0}^{1} 1-x\ \mathrm{d}\,x$$
$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\ \mathrm{d}\,x = \int_{-1}^{0}(x+1)\cos(n\pi x)\ \mathrm{d}\,x + \int_{0}^{1}(1-x)\cos(n\pi x)\ \mathrm{d}\,x$$
$$b_n = \int_{-1}^{0}(x+1)\sin(n\pi x)\ \mathrm{d}\,x + \int_{0}^{1}(1-x)\sin(n\pi x)\ \mathrm{d}\,x$$

10.3 Fourier Convergence Theorem

If f and f' are piecewise continuous, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right].$$

The Fourier series converges to $f(x)$ for all values of x where f is continuous, and it converges to $\frac{f(x^-)+f(x^+)}{2}$ at the points of discontinuity of f. Note that if it is continuous, then $f(x^-) = f(x^+)$, making the whole expression reduce to $f(x)$.

Example 10.10.

$$f(x+2L) = f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}$$

Step-1 Express $f(x)$ as a Fourier Series.

$$a_0 = L$$
$$a_n = 0$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}\,x$$
$$b_n = \frac{1}{L}\left[\int_{-L}^{0} 0\ \mathrm{d}\,x + \int_{0}^{L} L\sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}\,x\right] = \frac{1}{L}\cdot L\cdot\frac{L}{n\pi}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$
$$b_n = \frac{L}{n\pi}(1-\cos(n\pi)) = \begin{cases} \frac{2L}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$f(x) = \frac{L}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) = \frac{L}{2} + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{L}\right)$$

Now we want to use this Fourier series to find the value of the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

We do this by matching terms. First we need to make the sum go from $n = 0$ instead of $n = 1$

$$f(x) = \frac{L}{2} + \frac{2L}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi x}{L}\right)}{2n+1}$$

$$\text{When } n = 0, \ \sin(\ldots) = 1$$

$$\frac{\pi x}{L} = \frac{\pi}{2} \implies x = \frac{L}{2}$$

This also gets us -1 for odd values of n, and back to 1 for even values

$$f\left(\frac{L}{2}\right) = L = \frac{L}{2} + \frac{2L}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{2n+1} = \frac{L}{2} + \frac{2L}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\frac{L}{2} = \frac{2L}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \implies \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Definition 10.6 (Perseval's Equation)**.**

$$\frac{1}{L} \int_{-L}^{L} f(x)^2 \, \mathrm{d}\, x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{10.7}$$

Proof. Starting from the Fourier series definition, and multiplying by $f(x)$, we get

$$f(x)^2 = \frac{a_0}{2} f(x) + \sum_{n=1}^{\infty} \left[a_n f(x) \cos\left(\frac{n\pi x}{L}\right) + b_n f(x) \sin\left(\frac{n\pi x}{L}\right)\right]$$

Now integrate both sides w.r.t. x from $-L$ to L

$$\int_{-L}^{L} f(x)^2 \, \mathrm{d}\, x = \frac{a_0}{2} \int_{-L}^{L} f(x) \, \mathrm{d}\, x + \sum_{n=1}^{\infty} \left[a_n \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, \mathrm{d}\, x + b_n \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}\, x\right]$$

Now divide both sides by L.

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 \,\mathrm{d}\,x = \frac{a_0}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}\,x + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x + \frac{b_n}{2}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x\right]$$

Recall our definitions of a_0, a_n, and b_n

$$\begin{cases} a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}\,x \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x \end{cases}$$

These fit right into the previous equation, leaving us with

$$\frac{1}{L}\int_{-L}^{L} f(x)^2\,\mathrm{d}\,x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

□

10.4 Even and Odd Functions

A function f is even if $f(-x) = f(x)$, meaning it is symmetrical across the y-axis. A function f is odd if $f(-x) = -f(x)$.

1. If f and g are odd, then
   (a) $f \pm g$ is odd
   (b) $f \cdot g$ is even
   (c) $f \div g$ is even (if $g \neq 0$)
2. If f and g are even, then
   (a) $f \pm g$ is even
   (b) $f \cdot g$ is even

(c) $f \div g$ is even (if $g \neq 0$)

3. If f is even and g is odd

   (a) $f \pm g$ is neither even nor odd

   (b) $f \cdot g$ is odd

   (c) $f \div g$ is odd (if $g \neq 0$)

4. If $f \equiv 0$, then f is both an odd and even function

5. If f is even
$$\int_{-L}^{L} f(x)\,\mathrm{d}\,x = 2\int_{0}^{L} f(x)\,\mathrm{d}\,x$$

6. If f is odd
$$\int_{-L}^{L} f(x)\,\mathrm{d}\,x = 0$$

Definition 10.7 (Cosine Fourier Series)**.** Suppose f is an even function, f and f' are piecewise continuous, and f has a period $2L$

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}\,x = \frac{2}{L}\int_{0}^{L} f(x)\,\mathrm{d}\,x \\
a_n &= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x \\
b_n &= 0 \\
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

Definition 10.8. Suppose f is an odd function, f and f' are piecewise continuous, and f has a period $2L$

$$\begin{aligned}
a_0 &= 0 \\
a_n &= 0 \\
b_n &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x
\end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Example 10.11.

$$f(x) = x,\ -L < x < L;\ f(-L) = f(L) = 0$$

Find the Fourier series for f. Note that f is an odd function, so it is going to be a Fourier sine series.

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x$$

$$b_n = \frac{2L}{n\pi}\left[\sin(n\pi) - \cos(n\pi)\right] = \frac{2L}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$

Let's apply what we've learned to Perseval's equation

$$\frac{2}{L}\int_0^L f(x)^2\,\mathrm{d}\,x = \frac{a_0^2}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

For a Fourier cosine series, the equation becomes

$$\frac{2}{L}\int_0^L f(x)^2\,\mathrm{d}\,x = \frac{a_0^2}{2}\sum_{n=1}^{\infty}a_n^2,$$

and for a Fourier sine series, the equation becomes

$$\frac{2}{L}\int_0^L f(x)^2\,\mathrm{d}\,x = \sum_{n=1}^{\infty}b_n^2.$$

Example 10.12.

$$g(x) \equiv f(x),\ 0 \le x < L$$

Now we want to extend $g(x)$ as a periodic function. We can do this either as a sine, cosine, or mixed periodic function. First let's do cosine (even)

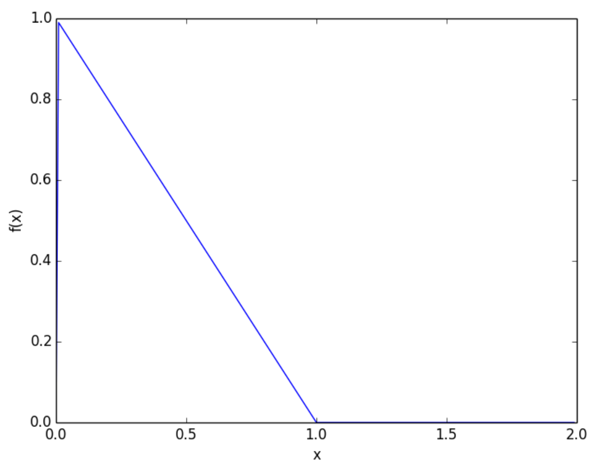
$$g(x) = \begin{cases} f(x), & 0 \le x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

As a sine periodic function, it is

$$g(x) = \begin{cases} f(x), & 0 \le x < L, \\ -f(-x), & -L < x < 0, \\ 0 & x \in \{0,\ L\}. \end{cases}$$

Example 10.13.

$$f(x) = \begin{cases} 1 - x, & 0 < x \le 1, \\ 0, & 1 < x \le 2. \end{cases}$$



Sketch an even periodic extension. $(L = 2)$

$$\begin{cases} 1 - x, & 0 < x \leq 1, \\ 0, & 1 < x \leq 2, \\ 1 + x, & -1 < x \leq 0, \\ 0 & -2 < x \leq -1. \end{cases}$$

Sketch an odd periodic extension.

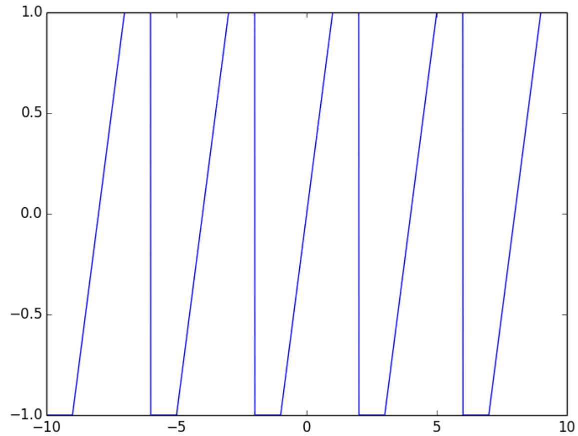
$$\begin{cases} 1 - x, & 0 < x \leq 1, \\ 0, & 1 < x \leq 2, \\ x - 1, & -1 < x \leq 0, \\ 0 & -2 < x \leq -1. \end{cases}$$

Example 10.14.

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

Show the sine extension for this function.

$$f(x) = \begin{cases} x, & -1 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ -1, & -2 \leq x < -1 \end{cases}$$



10.5 Separation of Variables; Heat Conduction in a Rod

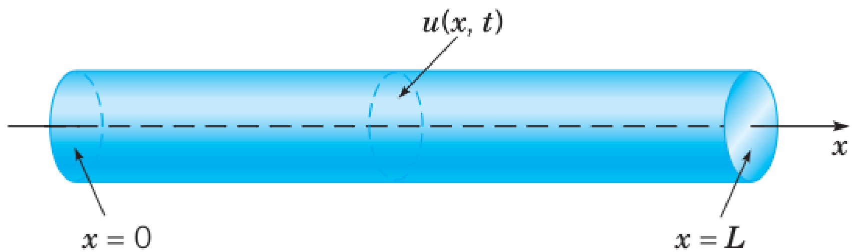


Figure 1: A heat-conducting solid bar.

- uniform cross section
- homogeneous

$$\alpha^2 U_{xx} = U_t, \ 0 < x < L, \ t > 0 \tag{10.8}$$

Set of assumptions

1. $U(x, 0) = f(x), \ 0 < x < L$
2. $U(0, t) = 0, \ U(L, t) = 0$

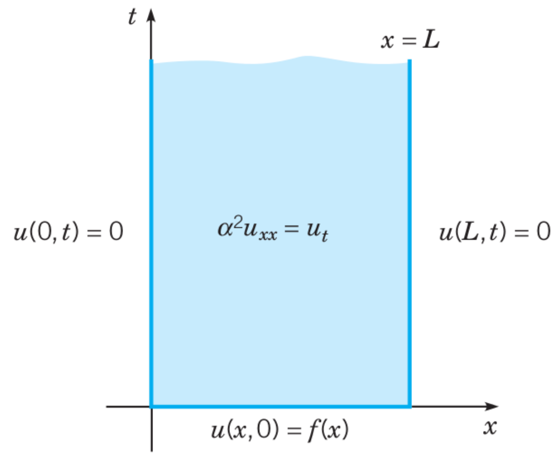


Figure 2: Boundary value problem for the heat conduction equation

Trivial case

$U(x,\, t) = 0$, then Equation 10.8 and assumption 1 are satisfied. To satisfy assumption 1, we need to have $f(x) = 0$ for $0 < x < L$.

Non-Trivial case

Let $U(x,\, t) = X(x)T(t)$.

$$U_{xx} = X''(x)T(t),\ U_t = X(x)T'(t)$$

$$\alpha^2 U_{xx} = U_t = \alpha^2 X''(x)T(t) = X(x)T'(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = \text{const} = -\lambda$$

The two must equal some constant, because X is only a function of x, and T is only a function of t, and in order for them to be equal, all of the x's and t's must cancel out from the division, leaving us with a constant.

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda \alpha^2 T = 0 \end{cases}$$

Now let's look at our conditions. Using assumption 2, we find that

$$0 = U(0,\, t) = X(0)T(t)$$

So either $X(0) = 0$ or $T(t) = 0$ for all t. If $T(t) = 0$, then $U(x, t) = 0$, which was the trivial case, so we are going to assume that $T(t) \neq 0$, meaning $X(0) = 0$.

$$0 = U(L, t) = X(L)T(t) \implies X(L) = 0$$

We now have the boundary value problem

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = 0, \; X(L) = 0, \end{cases}$$

which has solution

$$\begin{cases} X_n = \sin\left(\frac{n\pi x}{L}\right), \; n = 1, 2, 3, \ldots \\ \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, 3, \ldots \end{cases}$$

Now let's look at the BVP for T

$$T' + \left(\frac{n\pi\alpha}{L}\right)^2 T = 0$$

The characteristic equation is

$$r + \left(\frac{n\pi\alpha}{L}\right)^2 = 0 \implies r = -\left(\frac{n\pi\alpha}{L}\right)^2$$

Because this is a first-degree ODE with one root, we have

$$T_n = e^{rt} = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}.$$

Now we know that $U_n(x, t) = X_n T_n$, so we get eigenvector

$$U_n = e^{rt} = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \; n = 1, 2, 3, \ldots$$

Recall our first assumption

$$f(x) = U(x,\, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

This is a Fourier sine series, so we can compute c_n as

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}\, x.$$

Now we know that any linear combination of possible solutions is a solution, so our general solution is

$$U(x,\, t) = \sum_{n=1}^{\infty} c_n U_n(x,\, t)$$

Example 10.15 (Example 1 on p628)**.**

$$U(x,t) = ?,\ L = 50\text{cm}$$

$$U(x,0) = 20,\ U(0,t) = 0,\ U(50\text{cm},t) = 0$$

$$\alpha^2 U_{xx} = U_t$$

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{50}\right)^2 t} \sin\left(\frac{n\pi x}{50}\right)$$

$$c_n = \frac{2}{50}\int_0^L \underbrace{20}_{f(x)} \sin\left(\frac{n\pi x}{50}\right) = \frac{4}{5}\left[\frac{50}{n\pi}\cos\left(\frac{n\pi x}{50}\right)\right]_{50}^{0} = \frac{40}{n\pi}\left[1 - \cos(n\pi)\right]$$

$$c_n = \begin{cases} \dfrac{80}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$U(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{80}{n\pi} e^{-\left(\frac{n\pi\alpha}{50}\right)^2 t} \sin\left(\frac{n\pi x}{50}\right)$$

Example 10.16.

$$xU_{xx} + U_t = 0$$

Find the system of equations for $X(x)$ and $T(t)$.

$$U(x,t) = X(x)T(t)$$

$$U_{xx} = X''T, \; U_t = XT'$$

$$xX''T + XT' = 0 \implies xX''T = -XT' \implies \frac{xX''}{X} = -\frac{T'}{T} = -\lambda$$

$$\begin{cases} xX'' + \lambda X = 0 \\ T' - \lambda T = 0 \end{cases}$$

Example 10.17.

$$U_{xx} + (x+y)U_{yy} = 0$$

Not separable, because of the $(x+y)$ term.

Example 10.18.

$$tU_{xx} + xU_t = 0$$

Find the system of two equations

$$U(x,t) = X(x)T(t); \; \frac{\partial^2 U}{\partial x^2} = T(t)\frac{\mathrm{d}^2 X}{\mathrm{d}\,x^2}; \; \frac{\partial U}{\partial t} = X(x)\frac{\mathrm{d}\,T}{\mathrm{d}\,t}$$

$$tT(t)\frac{\mathrm{d}^2 X}{\mathrm{d}\,x^2} + xX(x)\frac{\mathrm{d}\,T}{\mathrm{d}\,t} = 0 \implies \frac{1}{xX(x)}\frac{\mathrm{d}^2 X}{\mathrm{d}\,x^2} = -\frac{1}{tT(t)}\frac{\mathrm{d}\,T}{\mathrm{d}\,t} = -\lambda$$

$$\begin{cases} X''(x) + \lambda x X(x) = 0 \\ T'(t) - \lambda t T(t) = 0 \end{cases}$$

Example 10.19.

$$[p(x)U_x]_x - r(x)U_{tt} = 0$$

Find the system of two equations

$$\frac{\partial}{\partial x}\left[p(x)T(t)\frac{\mathrm{d}\,X}{\mathrm{d}\,x}\right] - r(x)X(x)\frac{\mathrm{d}^2 T}{\mathrm{d}\,t^2} = T(t)\frac{\partial}{\partial x}\left[p(x)\frac{\mathrm{d}\,X}{\mathrm{d}\,x}\right] - r(x)X(x)\frac{\mathrm{d}^2 T}{\mathrm{d}\,t^2} = 0$$

$$T(t)\left[p(x)\frac{\mathrm{d}^2 X}{\mathrm{d}\,x^2} + \frac{\mathrm{d}\,p}{\mathrm{d}\,x}\frac{\mathrm{d}\,X}{\mathrm{d}\,x}\right] - r(x)X(x)\frac{\mathrm{d}^2 T}{\mathrm{d}\,t^2} = 0$$

$$T(t)\left[p(x)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}+\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}X}{\mathrm{d}x}\right]=r(x)X(x)\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} \implies \frac{p(x)\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}+\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}X}{\mathrm{d}x}}{r(x)X(x)}=\frac{1}{T(t)}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2}=-\lambda$$

$$\begin{cases} p(x)X''(x)+p'(x)X'(x)+\lambda r(x)X(x)=0 \\ T''(t)+\lambda T(t)=0 \end{cases}$$

Example 10.20.

$$\alpha^2\left[U_{rr}+\frac{1}{r}U_r+\frac{1}{r^2}U_{\theta\theta}\right]=U_t$$

Find the system of three equations

$$U(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

$$U_r=\Theta T R';\ U_{rr}=\Theta T R'';\ U_t=R\Theta T';\ U_{\theta\theta}=RT\Theta''$$

$$\alpha^2\left[\Theta T R''+\frac{1}{r}\Theta T R'+\frac{1}{r^2}RT\Theta''\right]=R\Theta T'$$

$$\alpha^2\frac{\Theta R''+\frac{1}{r}\Theta R'+\frac{1}{r^2}R\Theta''}{R\Theta}=\frac{T'}{T}=-\beta=\text{const.}$$

$$\alpha^2\left[\Theta R''+\frac{1}{r}\Theta R'+\frac{1}{r^2}R\Theta''\right]+\beta R\Theta=0$$

$$\alpha^2\left[\Theta R''+\frac{1}{r}\Theta R'\right]+\beta R\Theta=-\frac{1}{r^2}R\Theta''$$

$$\alpha^2\left[\frac{r^2R''+rR'}{R}\right]+r^2\beta=-\frac{\Theta''}{\Theta}=\gamma=\text{const.}$$

$$\begin{cases} \alpha^2r^2R''+\alpha^2rR'+(r^2\beta-\gamma)R=0 \\ \Theta''+\gamma\Theta=0 \\ T'+\beta T=0 \end{cases}$$

This could have been simpler if we'd simplified after dividing by $R\Theta$, and if we grouped the α term with the T's. That way the equation for R wouldn't have 3 constants.

Example 10.21. Find the solution of the heat conduction problem.

$$\begin{cases} 100U_{xx} = U_t, & 0 < x < 1,\ t > 0, \\ U(0,t) = 0,\ U(1,t) = 0, & t > 0, \\ U(x,0) = \underbrace{\sin(2\pi x) - \sin(5\pi x)}_{f(x)}, & 0 \le x \le 1; \end{cases}$$

Here, $L = 1$, so we can express c_n as

$$c_n = \frac{2}{1}\int_0^1 [\sin(2\pi x) - \sin(5\pi x)]\sin(n\pi x)\ \mathrm{d}\,x$$

$$c_n = 2\left[\int_0^1 \sin(2\pi x)\sin(n\pi x)\ \mathrm{d}\,x - \int_0^1 \sin(5\pi x)\sin(n\pi x)\ \mathrm{d}\,x\right]$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$c_n = \int_0^1 \cos((2-n)\pi x) - \cos((2+n)\pi x)\ \mathrm{d}\,x + \int_0^1 \cos((5-n)\pi x) - \cos((5+n)\pi x)\ \mathrm{d}\,x$$

When n is neither 2 nor 5, the whole expression is zero, because the integral will be the sine of an integral multiple of π, so we just have

$$c_2 = 2\int_0^1 \sin^2(2\pi x)\ \mathrm{d}\,x$$

$$c_5 = -2\int_0^1 \sin^2(5\pi x)\ \mathrm{d}\,x$$

10.6 Other Heat Conduction Problems

Previously, we had $U(0,t) = U(L,t) = 0$, so we had homogeneous boundary conditions.

$$\lim_{t\to\infty} U(x,t) = 0$$

But what if we had something other than 0 at the boundaries? Then we would have nonhomegeneous boundary conditions.

$$\alpha^2 U_{xx} = U_t,\ 0 < x < L,\ t > 0$$
$$U(x,0) = f(x)$$
$$U(0,t) = T_1,\ U(L,t) = T_2$$

As time goes by, there will be a steady temperature, independent of t and the initial condition. Let's call it $V(x)$. Since it is not dependent on t, that means

$$\alpha^2 V_{xx} = 0 \implies V_{xx} = 0 \implies V_x = c_1 \implies V = c_1 x + c_2$$

Recall boundary conditions, $V(0) = T_1$, $V(L) = T_2$.

$$V(0) = T_1 = c_2$$
$$V(L) = T_2 = c_1 L + T_1 \implies c_1 = \frac{T_2 - T_1}{L}$$
$$V(x) = \frac{T_2 - T_1}{L} x + T_1$$

This gives us homogeneous boundary conditions if we just set $T_1 = T_2 = 0$.

$$U(x,t) := W(x,t) + V(x)$$

In the homogeneous case, we had $V(x) = 0$.

$$\alpha^2 (W + V)_{xx} = (W + V)_t$$
$$\alpha^2 W_{xx} = W_t \text{ because } V_{xx} = V_t = 0$$
$$W(x,0) = U(x,0) - V(x) = f(x) - V(x)$$
$$W(0,t) = U(0,t) - V(0) = T_1 - T_1 = 0$$
$$W(L,t) = U(L,t) - V(L) = T_2 - T_2 = 0$$

So we have recreated the homogeneous solution for $U(x,t)$, but for $W(x,t)$, so

$$W(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$
$$c_n = \frac{2}{L}\int_0^L [f(x) - V(x)] \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}\,x$$
$$U(x,t) = W(x,t) + V(x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) + \frac{T_2 - T_1}{L}x + T_1,$$
$$\text{where } c_n = \frac{2}{L}\int_0^L \left[f(x) - \frac{T_2 - T_1}{L}x - T_1\right] \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}\,x$$

Example 10.22.

$$\begin{cases} U_{xx} = U_t,\ 0 < x < 30, & t > 0 \\ U(0,t) = 20,\ U(30,t) = 50, & t > 0 \\ U(x,0) = 60 - 2x = f(x), & 0 < x < 30 \end{cases}$$

$$c_n = \frac{2}{30}\int_0^{30} \left[60 - 2x - \frac{50-20}{30}x - 20\right] \sin\left(\frac{n\pi x}{30}\right) \,\mathrm{d}\,x$$
$$c_n = \frac{1}{15}\int_0^{30} [-3x + 40] \sin\left(\frac{n\pi x}{30}\right) \,\mathrm{d}\,x$$
$$\text{IBP: } u = -3x + 40,\ u' = -3,\ v' = \sin\left(\frac{n\pi x}{30}\right),\ v = -\frac{30}{n\pi}\cos\left(\frac{n\pi x}{30}\right)$$
$$c_n = \frac{1}{50}\left(\left[-\frac{30(-3x+40)}{n\pi}\cos\left(\frac{n\pi x}{30}\right)\right]_0^{30} - \int_0^{30} \frac{90}{n\pi}\cos\left(\frac{n\pi x}{30}\right) \,\mathrm{d}\,x\right)$$

In general, for a cylinder with insulated end points, the following is true

$$\begin{cases} \alpha^2 U_{xx} = U_t \\ U(x,0) = f(x) \\ U_x(0,t) = 0,\ U_x(L,t) = 0 \end{cases}$$

Let $U(x,t) = X(x)T(t)$

$$U_x = X'T,\ U_{xx} = X''T,\ U_t = XT'$$

$$\alpha^2 X''T = XT' \implies \frac{X''}{X} = \frac{T'}{\alpha^2 T} := -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda\alpha^2 T = 0 \end{cases}$$

$$\begin{cases} U_x(0,t) = X'(0)T(t) = 0 \\ U_x(L,t) = X'(L)T(t) = 0 \end{cases}$$

We can't have $T(t) = 0$ for all t, so $X'(0) = X'(L) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0;\ X'(L) = 0 \end{cases}$$

$$r^2 + \lambda = 0 \implies r = \sqrt{-\lambda}$$

Case 1: $\lambda > 0$

$$r = \pm\sqrt{\lambda}\imath$$

$$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$X'(x) = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2\cos(\sqrt{\lambda}x)$$

$$X'(0) = 0 = c_2\sqrt{\lambda} \implies c_2 = 0$$

$$X'(x) = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}x)$$

$$X'(L) = 0 = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}L) \implies 0 = c_1\sin(\sqrt{\lambda}L)$$

$$\sin(\sqrt{\lambda}L) = 0 \implies \sqrt{\lambda}L = n\pi,\ n \in \mathbb{Z}^+ \implies \lambda_n = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{Z}^+$$

Eigenfunction: $X_n(x) = c_1\cos\left(\frac{n\pi x}{L}\right)$

any constant multiple of an Eigenfunction is an Eigenfunction, so:

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

Now that we have λ for this case, we substitute into $T(t)$'s D.E.

$$T' + \lambda\alpha^2 T = 0 \implies T' + \left(\frac{n\pi}{L}\right)^2\alpha^2 T = 0$$

$$r + \left(\frac{n\pi\alpha}{L}\right)^2 = 0$$

$$T_n(t) = c_1 e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$U_n(x,t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

Case 2: $\lambda < 0$

$$r^2 + \lambda = 0 \implies r^2 = -\lambda := \mu \implies r = \pm\sqrt{\mu}$$

$$X(x) = c_1 e^{\sqrt{\mu}x} + c_2 e^{-\sqrt{\mu}x}$$

$$X'(x) = \sqrt{\mu} c_1 e^{\sqrt{\mu}x} - \sqrt{\mu} c_2 e^{-\sqrt{\mu}x}$$

$$X'(0) = 0 = \sqrt{\mu} c_1 - \sqrt{\mu} c_2 \implies c_1 = c_2$$

$$X'(L) = 0 = \sqrt{\mu} c_1 e^{\sqrt{\mu}L} - c_1 \sqrt{\mu} e^{-\sqrt{\mu}L}$$

$$0 = \sqrt{\mu} c_1 \left(\frac{e^{2\sqrt{\mu}L} - 1}{e^{\sqrt{\mu}L}}\right)$$

$$\sqrt{\mu} \neq 0,\ e^{\sqrt{\mu}L} \neq 0 \implies c_1\left(e^{2\sqrt{\mu}L} - 1\right) = 0,\ e^{2\sqrt{\mu}L} - 1 \neq 0 \implies c_1 = 0 \text{ (trivial solution)}$$

Case 3: $\lambda = 0$

$$X'' = 0 \implies X' = c_1 \implies X(x) = c_1 x + c_2$$

$$X'(0) = 0 \implies c_1 = 0 \implies X(x) = c_2$$

$$T'(t) = 0 \implies T(t) = c_3$$

$$U(x,t) = X(x)T(t) = c_2 c_3 = c$$

$$U_0(x,t) = c$$

$$U(x,t) = kc + c_n U_n(x,t) \text{ (linear combination)}$$

$$U(x,t) = \frac{c_0}{2} + c_n U_n(x,t)$$

$$U(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}\,x,\ n = 0,\ 1,\ 2,\ 3,\ \ldots$$

10.7 The Wave Equation: Vibrations of an Elastic String

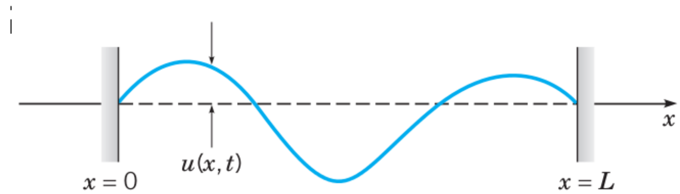


Figure 3: A vibrating string.

$$\begin{cases} a^2 U_{xx} = U_{tt} \\ U(0,t) = 0,\ U(L,t) = 0 \\ U(x,0) = f(x) \\ U_t(x,0) = g(x) \text{ (initial velocity)} \\ f(0) = f(L) = g(0) = g(L) = 0 \end{cases}$$

a depends on:

1. tension of the string
2. mass per unit length of the string

Case 1: initial velocity is zero

$$U(x,t) := X(x)T(t) \implies U_{xx} = X''T,\ U_{tt} = XT''$$

$$a^2 X''T = XT'' \implies \frac{X''}{X} = \frac{T''}{a^2 T} := -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + \lambda a^2 T = 0 \end{cases}$$

$$U(0,t) = U(L,t) = 0 \implies X(0) = X(L) = 0$$

$$U_t(x,0) = 0 \implies T'(0) = 0$$

We have already solved for $X(x)$ in previous problems, so we won't show the work again.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \; X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

To solve for $T(t)$:

$$\begin{gathered}
T'' + \left(\frac{n\pi a}{L}\right)^2 T = 0 \\
r^2 + \left(\frac{n\pi a}{L}\right)^2 = 0 \implies r = \pm\frac{n\pi a}{L}\imath \\
T(t) = c_1 \cos\left(\frac{n\pi a}{L}t\right) + c_2 \sin\left(\frac{n\pi a}{L}t\right), \; T'(0) = 0 \\
T'(t) = -c_1 \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L}t\right) + c_2 \frac{L}{n\pi a} \cos\left(\frac{n\pi a}{L}t\right) \\
T'(0) = 0 = c_1 \sin(0) + c_2 \cos(0) \implies c_2 = 0 \\
T_n(t) = \cos\left(\frac{n\pi a}{L}t\right) \implies U_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right) \\
U(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L}t\right)
\end{gathered}$$

This satisfies our condition $U_t(x,0) = 0$, because when we take the derivative, the cosine term becomes a sine term, which is zero when $t = 0$, making the entire expression zero. Let's check the condition $U(x,0) = f(x)$, and solve for c_n.

$$\begin{gathered}
f(x) = U(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \\
c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \,\mathrm{d}\, x
\end{gathered}$$

Example 10.23.

$$\begin{cases} 4U_{xx} = U_{tt}, \; 0 < x < 30, \; t > 0 \\ U(x,0) = f(x) = \begin{cases} \frac{x}{10}, & 0 \le x \le 10 \\ \frac{30-x}{20}, & 10 < x \le 30 \end{cases} \end{cases}$$

If we can pattern match to find $f(x)$, a, and L, we can just use our solution from before. $f(x)$ is given, a^2 is 4, so $a = 2$, and $L = 30$.

$$c_n = \frac{2}{30}\int_0^{30} f(x)\sin\left(\frac{n\pi x}{30}\right)\,\mathrm{d}\,x$$

$$c_n = \frac{1}{15}\left[\int_0^{10}\frac{x}{10}\sin\left(\frac{n\pi x}{30}\right)\,\mathrm{d}\,x + \int_{10}^{30}\frac{30-x}{20}\sin\left(\frac{n\pi x}{30}\right)\,\mathrm{d}\,x\right] = \left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)$$

$$U(x,t) = \sum_{n=1}^{\infty}\left(\frac{3}{n\pi}\right)^2\sin\left(\frac{n\pi}{3}\right)\sin\left(\frac{n\pi x}{30}\right)\cos\left(\frac{n\pi}{15}\right)$$

Case 2: initial velocity is non-zero, but $f(x) \equiv 0$

$$\begin{cases} a^2U_{xx} = U_{tt} \\ U(0,t) = 0,\ U(L,t) = 0 \\ U(x,0) = f(x) = 0,\ U_t(x,0) = g(x) \end{cases}$$

(In Case 1, $g(x) \equiv 0$)

$$U(x,t) = X(x)T(t)$$

$$\begin{cases} X'' + \lambda X = 0; \quad X(0) = X(L) = 0 \\ T'' + a^2\lambda T = 0; \quad T(0) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2,\ X_n = \sin\left(\frac{n\pi x}{L}\right)$$

$$T'' + \left(\frac{n\pi a}{L}\right)^2 T = 0;\ T(0) = 0$$

$$T_n = c_1\cos\left(\frac{n\pi a}{L}t\right) + c_2\sin\left(\frac{n\pi a}{L}t\right)$$

$$T_n(0) = 0 = c_1 \implies T_n(t) = \sin\left(\frac{n\pi a}{L}t\right)$$

$$U_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a}{L}t\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a}{L}t\right)$$

$$c_n = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x$$

but $U_t(x,0) = g(x)$, so we need to satisfy that

$$U_t(x,t) = \frac{n\pi a}{L}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a}{L}t\right)$$

$$U_t(x,0) = \frac{n\pi a}{L}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = c_n\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x$$

So we can summarize Case 2 as follows

$$\begin{cases} a^2U_{xx} = U_{tt} \\ U(0,t) = 0,\ U(L,t) = 0 \\ U(x,0) = 0,\ U_t(x,0) = g(x) \\ U(x,t) = \displaystyle\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a}{L}t\right) \\ c_n = \dfrac{2}{n\pi a}\displaystyle\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}\,x \end{cases}$$

Case 3: General Case

Let $U(x,t) = V(x,t) + W(x,t)$, where $V(x,t)$ is a solution to the PDE in Case 1, $W(x,t)$ is a solution to the PDE in Case 2, and as a result, $U(x,t)$ is the general solution to the wave equation. Let's prove that it is actually the general solution.

$$U_{xx} = V_{xx} + W_{xx};\ U_{tt} = V_{tt} + W_{tt}$$

$$a^2U_{xx} - U_{tt} = a^2\left(V_{xx} + W_{xx}\right) - \left(V_{tt} + W_{tt}\right) = 0$$

$$\underbrace{a^2V_{xx} - V_{tt}}_{0} + \underbrace{a^2W_{xx} - W_{tt}}_{0} = 0$$

$$U(0,t) = V(0,t) + W(0,t) = 0 + 0 = 0$$

$$U(L,t) = V(L,t) + W(L,t) = 0 + 0 = 0$$

$$U(x,0) = V(x,0) + W(x,0) = f(x) + 0 = f(x)$$

$$U_t(x,0) = V_t(x,0) + W_t(x,0) = 0 + g(x) = g(x)$$

So every condition has been satisfied. We basically chose the two cases so that we had one with an initial displacement but no initial velocity, and one with an initial velocity but no initial displacement, so we could combine them into the general case with an initial displacement and an initial velocity (which *can* still be zero).

$$\begin{cases} U(x,t) = \displaystyle\sum_{n=1}^{\infty} \left[c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a}{L} t\right) + k_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a}{L} t\right)\right] \\ U(x,t) = \displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[c_n \cos\left(\frac{n\pi a}{L} t\right) + k_n \sin\left(\frac{n\pi a}{L} t\right)\right] \\ c_n = \displaystyle\frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}\, x \\ k_n = \displaystyle\frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}\, x \end{cases}$$

10.8 Laplace's Equation

$$\Delta U = 0$$

$$\text{In 2D: } \Delta U = U_{xx} + U_{yy} = 0$$

$$\text{In 3D: } \Delta U = U_{xx} + U_{yy} + U_{zz} = 0$$

$$\text{Hessian of } U\text{: } D^2 U = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}$$

$$\text{Tr}\left[D^2 U\right] = \Delta U = 0$$

$$D^2 U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & \dots & U_{1n} \\ U_{21} & U_{22} & U_{23} & \dots & U_{2n} \\ U_{31} & U_{32} & U_{33} & \dots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{n1} & U_{n2} & U_{n3} & \dots & U_{nn} \end{bmatrix}$$

Laplace equation in different dimensions

$$1\text{D: } \alpha^2 U_{xx} = U_t$$

$$2\text{D: } \alpha^2 (U_{xx} + U_{yy}) = U_t$$

$$\alpha^2 \Delta U = U_t \text{ (Heat Equation)}$$

Dirchelt problem: $\Delta U = 0$, with certain conditions on the boundary.

Neumann problem: $\Delta U = 0$, boundary conditions depend on some derivatives of U.

Example 10.24 (Dirichlet problem on a rectangle)**.** $\Delta U = 0$

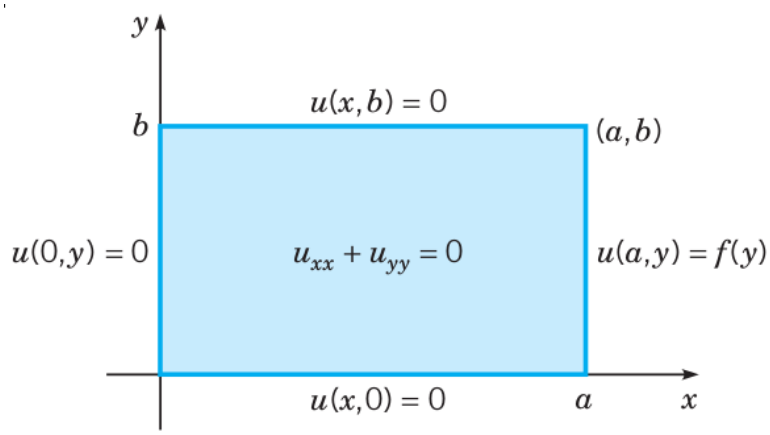


Figure 4

$$U(x,y) = X(x)Y(y)$$

$$\Delta U = (X''Y + XY'') = 0 \implies X''Y = -XY'' \implies \frac{X''}{X} = -\frac{Y''}{Y} := -\lambda$$

$$U(x,0) = 0 = X(x)Y(0) \implies Y(0) = 0$$

$$U(x,b) = 0 = X(x)Y(b) \implies Y(b) = 0$$

$$U(0,y) = 0 = X(0)Y(y) \implies X(0) = 0$$

$$\begin{cases} X'' + \lambda X = 0; \quad X(0) = 0 \\ Y'' - \lambda Y = 0; \quad Y(0) = Y(b) = 0 \end{cases}$$

$$Y_n = \sin\left(\frac{n\pi y}{b}\right), \ \lambda_n = \left(\frac{n\pi}{b}\right)^2$$
$$X'' - \left(\frac{n\pi}{b}\right)^2 X = 0; \ r^2 - \left(\frac{n\pi}{b}\right)^2 = 0 \implies r = \pm\frac{n\pi}{b}$$
$$X(x) = c_1 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x}$$
$$X(0) = 0 = c_1 + c_2 \implies c_1 = -c_2$$
$$X(x) = -c_2 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x} = c_2\left(e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right) = 2\,c_2 \sinh\left(\frac{n\pi}{x}b\right)$$
$$X_n(x) = e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x} = \sinh\left(\frac{n\pi}{x}b\right)$$
$$U_n(x,y) = X_n(x)Y_n(y) = \left[e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right]\sin\left(\frac{n\pi}{b}y\right)$$
$$U(x,y) = \sum_{n=1}^{\infty} c_n\left[e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right]\sin\left(\frac{n\pi}{b}y\right)$$

This satisfies every condition, except for $U(a, y) = f(y)$, so we have to use the condition to find c_n.

$$U(a,y) = f(y) = \sum_{n=1}^{\infty} c_n\left[e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right]\sin\left(\frac{n\pi}{b}y\right)$$
$$\tilde{c}_n = c_n\left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)\,\mathrm{d}\,y$$

Example 10.25 (Dirichlet problem on a rectangle (2))**.** This time, we change the boundary condition $U(x, b) = 0$ to $U(x, b) = f(x)$, and $U(a, y) = f(y)$ to $U(a, y) = 0$.

Example 10.26 (Dirchlet problem on a circle of radius a)**.**

$$\begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 \\ U(0,\theta) \text{ is not a boundary condition.} \\ U(a,\theta) = f(\theta) \end{cases}$$

Solution $U(r, \theta)$ needs to be bounded, and periodic in θ, with period 2π.

$$U(r,\theta) = R(r)\Theta(\theta)$$
$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}=0$$
$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}:=\lambda$$
$$\begin{cases} r^2\frac{R''}{R}+r\frac{R'}{R}=\lambda \\ \frac{\Theta''}{\Theta}=-\lambda \end{cases}$$
$$\begin{cases} r^2R''+rR'-\lambda R=0 \\ \Theta''+\lambda\Theta=0 \end{cases}$$
$$\Theta''+\lambda\Theta=0 \implies m^2+\lambda=0 \implies m=\pm\sqrt{-\lambda}$$

Case 1: $\lambda<0$

$$m=\pm\sqrt{-\lambda}:=\pm\sqrt{\mu} \implies \Theta(\theta)=c_1e^{-\sqrt{\mu}\theta}+c_2e^{\sqrt{\mu}\theta}$$

If c_1 and c_2 are non-zero, we have an exponential function, which cannot be periodic. Therefore $c_1=c_2=0$, and we have a trivial case.

Case 2: $\lambda=0$

$$\Theta''=0 \implies \Theta'=c_1 \implies \Theta=c_1\theta+c_2$$

If c_1 is non-zero, we will have a linear, and therefore non-periodic function. However, if c_2 is non-zero, we will have a constant function, which *is* periodic, with any period. So $\Theta(\theta)=c_2$, or since any linear combination is also an eigenfunction, we can just say $\Theta(\theta)=1$. Now let's study $R(r)$, with $\lambda=0$, $r^2R''+rR'=0$. Any time we have $x^2y''(x)+xy'(x)=0$, we have an Euler equation. So here we make the substitution $x:=\ln r$.

$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}=0$$
$$\frac{\mathrm{d}R}{\mathrm{d}x}=\frac{\mathrm{d}R}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}x}$$
$$\text{since } x=\ln r,\ 1=\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}x} \implies \frac{\mathrm{d}r}{\mathrm{d}x}=r$$
$$\text{so } \frac{\mathrm{d}R}{\mathrm{d}x}=r\frac{\mathrm{d}R}{\mathrm{d}r}$$
$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2}+\frac{\mathrm{d}R}{\mathrm{d}x}=0$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} = \frac{\mathrm{d}}{\mathrm{d}\, x}\left[\frac{\mathrm{d}\, R}{\mathrm{d}\, x}\right] = \frac{\mathrm{d}}{\mathrm{d}\, r}\left[\frac{\mathrm{d}\, R}{\mathrm{d}\, x}\right]\frac{\mathrm{d}\, r}{\mathrm{d}\, x} = \frac{\mathrm{d}}{\mathrm{d}\, r}\left[\frac{\mathrm{d}\, R}{\mathrm{d}\, r} r\right] r = r\left[\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} r + \frac{\mathrm{d}\, R}{\mathrm{d}\, r}\right]$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} = r^2\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} + r\frac{\mathrm{d}\, R}{\mathrm{d}\, r} \implies r^2\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} = \frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} - r\frac{\mathrm{d}\, R}{\mathrm{d}\, r}$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} - r\frac{\mathrm{d}\, R}{\mathrm{d}\, r} + r\frac{\mathrm{d}\, R}{\mathrm{d}\, r} = 0 \implies \frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} = 0$$

$$R''(x) = 0 \implies R'(x) = c_1 \implies R(x) = c_1 x + c_2$$

$$x = \ln r \implies R(r) = c_1 \ln r + c_2$$

But remember that it has to be bounded, and as r approaches zero, $\ln r$ approaches negative infinity, which we can't have. So we must make $c_1 = 0$ making $R(r) = c_2$, or $R(r) = 1$. So we have one solution, $U_0(r, \theta) = 1$.

Case 3: $\lambda > 0$

$$m = \pm\sqrt{\lambda}\imath \implies \Theta(\theta) = c_1 \cos(\sqrt{\lambda}\theta) + c_2 \sin(\sqrt{\lambda}\theta) \implies \sqrt{\lambda} = n \in \mathbb{N}$$

$$\Theta_n(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$$

$$r^2 R'' + rR' - \lambda R = r^2 R'' + rR' - n^2 R = 0$$

Now let's reuse our Euler substitutions from before

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} = r^2\frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} + r\frac{\mathrm{d}\, R}{\mathrm{d}\, r} = r^2 R'' + rR'$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\, x^2} - n^2 R = 0$$

Now that we've gotten rid of the r's, we have R as a function of x, so we can just take the characteristic equation

$$m^2 - n^2 = 0 \implies m = \pm n$$

$$R(x) = c_1 e^{-nx} + c_2 e^{nx}$$

$$R(r) = c_1 e^{-n \ln r} + c_2 e^{n \ln r} = c_1 r^{-n} + c_2 r^n = \frac{c_1}{r^n} + c_2 r^n$$

Recall that this must be bounded on the circle, including at radius 0. If we substitute $r = 0$, we get division by zero, unless we set $c_1 := 0$, so we must do that, giving us $R(r) = c_2 r^n$, or simply $R_n(r) = r^n$. Now we multiply R_n and Θ_n to get U_n

$$U_n(r,\theta) = r^n c_1 \sin(n\theta) + r^n c_2 \cos(n\theta),\ n \in \mathbb{N}\,.$$

Now take all linear combinations to get the general solution

$$U(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[c_n r^n \cos(n\theta) + k_n r^n \sin(n\theta)\right].$$

Now let's look at our boundary conditions

$$U(a,\theta) = f(\theta) - \frac{c_0}{2} + \sum_{n=1}^{\infty} \left[c_n a^n \cos(n\theta) + k_n a^n \sin(n\theta)\right]$$
$$c_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\ \mathrm{d}\,\theta$$
$$k_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\ \mathrm{d}\,\theta$$

11 Boundary Value Problems and Sturm–Liouville Theory

11.1 The Occurence of Two-Point Boundary Value Problems

$$\left[p(x)U_x\right]_x - q(x)U = r(x)U_t$$

$$U_x(0,t) - h_1 U(0,t) = 0$$

$$U_x(L,t) + h_2 U(L,t) = 0$$

When $p(x)$ is constant, say $p(x) \equiv \alpha^2$, we have a special case, where

$$\left[p(x)U_x\right]_x = \alpha^2 U_{xx}.$$

If we have no source of heat (or whatever quantity we're measuring), then $q(x)U = 0$. And if $r(x)$ is constant, say $r(x) \equiv 1$, then we have a special case where $r(x)U_t = U_t$. So we have the heat equation from the previous chapter

$$\alpha^2 U_{xx} = U_t.$$

Now let's try solving this equation with separation of variables

$$[PX'T]_x - q(x)XT = r(x)XT'$$

Now divide by $r(x)XT$.

$$\frac{[PX'T]_x}{r(x)XT} - \frac{q(x)}{r(x)} = \frac{[PX']_x}{r(x)X} - \frac{q(x)}{r(x)} = \frac{T'}{T} := -\lambda$$

$$\begin{cases} T' + \lambda T = 0 \\ [PX']_x - q(x)X + \lambda + \lambda r(x)X = 0 \end{cases}$$

Example 11.1 (Problem 11)**.** Consider the general linear homogenous second order equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

We seek an integrating factor $\mu(x)$ such that, upon multiplying the equation by $\mu(x)$, we can write the resulting equation in the form

$$[\mu(x)P(x)y'(x)]' + \mu(x)R(x)y = 0$$

(a) By equating coefficients of y' in the previous two equations, show that μ must be a solution of

$$P(x)\mu'(x) = (Q(x) - P'(x))\mu(x) = 0$$

We start by taking the second equation, and evaluating the derivative in brackets using the product rule

$$[\mu(x)P(x)y'(x)]' = \mu'(x)P(x)y'(x) + \mu(x)\left[P'(x)y'(x) + P(x)y''(x)\right]$$

$$[\mu(x)P(x)y'(x)]' + \mu(x)R(x)y = \mu'(x)P(x)y'(x) + \mu(x)\left[P'(x)y'(x) + P(x)y''(x)\right] + \mu(x)R(x)y(x) = 0$$

$$0 = y'(x)\underbrace{(\mu'(x)P(x) + \mu(x)P'(x))}_{\mu(x)Q(x)} + \mu(x)P(x)y''(x) + \mu(x)R(x)y(x)$$

$$0 = \mu(x)P(x)y''(x) + \mu(x)Q(x)y'(x) + \mu(x)R(x)y(x)$$

(b) Solve the previous equation, and thereby show that

$$\mu(x) = \frac{1}{P(x)} \exp \int_{x_0}^{x} \frac{Q(s)}{P(s)} \,\mathrm{d}\, s.$$

First, note that

$$\frac{f'(x)}{f(x)} = [\ln[f(x)]]'.$$

Now take the equation, and divide throughout by $P(x)\mu(x)$. Then convert to the natural log form

$$\frac{Q(x)}{P(x)} = \frac{\mu'(x)}{\mu(x)} + \frac{P'(x)}{P(x)} = [\ln(\mu(x))]' + [\ln(P(x))]'$$

$$\int_{x_0}^{x} \frac{Q(s)}{P(s)} \,\mathrm{d}\, s = \ln(\mu(x)) + \ln(P(x)) = \ln(\mu(x)P(x))$$

$$\mu(x)P(x) = \exp \int_{x_0}^{x} \frac{Q(s)}{P(s)} \,\mathrm{d}\, s \implies \mu(x) = \frac{1}{P(x)} \exp \int_{x_0}^{x} \frac{Q(s)}{P(s)} \,\mathrm{d}\, s$$

Example 11.2.

$$y'' - 2xy' + \lambda y = 0$$

So $P(x) \equiv 1$, $Q(x) = -2x$, and $R(x) \equiv \lambda$.

$$\mu(x) = \frac{1}{1} \exp \int_{x_0}^{x} \frac{-2s}{1} \,\mathrm{d}\, s = \exp\left[-s^2\right]_{x_0}^{x} = \exp\left[x_0^2 - x^2\right] = e^{x_0^2} e^{-x^2} = c_1 e^{-x^2}$$

Any constant multiple of an integrating factor is also an integrating factor, so we can just set $c_1 := 1$.

Example 11.3.

$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$

So we have $P(x) = (1-x^2)$, $Q(x) = -x$, and $R(x) \equiv \alpha^2$.

$$\mu(x) = \frac{1}{1-x^2} \exp \int_{x_0}^{x} \frac{-s}{1-s^2} \, \mathrm{d}\, s = \frac{1}{1-x^2} \exp \left[\ln(s^2-1)/2 \right]_{x_0}^{x} = \frac{1}{2-2x^2} \exp \left[\ln(x^2-1) - \ln(x_0^2-1) \right]$$

$$\mu(x) = \frac{1}{1-x^2}\sqrt{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

Example 11.4.

$$\begin{cases} y'' - 2y' + (1+\lambda)y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

Doing things the way we did earlier, we would take the characteristic equation

$$r^2 - 2r + (1+\lambda) = 0$$

$$r = 1 \pm \sqrt{\lambda}\imath$$

$$y(x) = c_1 e^x \cos(\sqrt{\lambda}x) + c_2 e^x \sin(\sqrt{\lambda}x)$$

But we can do this using techniques from Chapter 10.

$$y = s(x)u(x)$$

Find $s(x)$ such that the differential equation has no u' term.

$$y' = s'u + su'$$

$$y'' = s''u + s'u + s'u' + su''$$

11.2 Sturm–Liouville Boundary Value Problems

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0; \quad \begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0, \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \\ 0 < x < 1 \end{cases}$$

Observe: for $p(x) = 1$, $q(x) = 0$, $r(x) = 1$: $y'' + \lambda y = 0$.

Definition 11.1 (Lagrange's Identity)**.**

$$L[u] := -[p(x)u']' + q(x)u \tag{11.1}$$

It follows that in the case of the problem we were just working on, $L[u] = \lambda r(x)u(x)$.

Lagrange's identity states that:

$$\int_0^1 (L[u]v - uL[v]) \, \mathrm{d}\, x = -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 \tag{11.2}$$

Let us derive it

$$\begin{aligned} \int_0^1 L[u]v \, \mathrm{d}\, x &= \int_0^1 [-(pu')'v + quv] \, \mathrm{d}\, x \\ &= \int_0^1 [-(pu')'v] \, \mathrm{d}\, x + \int_0^1 quv \, \mathrm{d}\, x \end{aligned}$$

Now let's use IBP with v as the function we differentiate and $-(pu')' \, \mathrm{d}\, x$ as the function we integrate. When we differentiate v we get v', and when we integrate the other term we get $-(pu')$.

$$\int_0^1 L[u]v \, \mathrm{d}\, x = -vpu'\Big|_0^1 + \int_0^1 pu'v' \, \mathrm{d}\, x + \int_0^1 quv \, \mathrm{d}\, x$$

Looking at what we're working towards, we have the $pu'v$ term, now we need to get the puv' term, but we have a $pu'v'$ term, so we use IBP again, this time with $u' \, \mathrm{d}\, x$ as the function we integrate, and pv' as the function we differentiate. The two functions become u after integrating, and $(pv')'$

after differentating.

$$\begin{aligned}
\int_0^1 L[u]v \, \mathrm{d}\, x &= -vpu'\Big|_0^1 + pv'u\Big|_0^1 - \int_0^1 (pv')'u \, \mathrm{d}\, x + \int_0^1 quv \, \mathrm{d}\, x \\
&= -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 + \int_0^1 \underbrace{-u(pv')' + quv}_{uL[v]} \, \mathrm{d}\, x \\
&= -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 + \int_0^1 uL[v] \, \mathrm{d}\, x \\
\int_0^1 (L[u]v - uL[v]) \, \mathrm{d}\, x &= -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1
\end{aligned}$$

Now suppose $\alpha_2 \neq 0$ and $\beta_2 \neq 0$. Let us evaluate the bounds on the integral in Laplace's identity

$$\int_0^1 (L[u]v - uL[v]) \, \mathrm{d}\, x = -p(1)[u'(1)v(1) - u(1)v'(1)] + p(0)[u'(0)v(0) - u(0)v'(0)]$$

$$\begin{aligned}
\alpha_1 u(0) + \alpha_2 u'(0) &= 0 \\
\alpha_1 v(0) + \alpha_2 v'(0) &= 0 \\
\beta_1 u(1) + \beta_2 u'(1) &= 0 \\
\beta_1 v(1) + \beta_2 v'(1) &= 0 \\
-\frac{\alpha_1}{\alpha_2} u(0) &= u'(0) \\
-\frac{\alpha_1}{\alpha_2} v(0) &= v'(0)
\end{aligned}$$

Now we can substitute $u'(0)$ and $v'(0)$ into the evaluated integral

$$\int_0^1 (L[u]v - uL[v]) \, \mathrm{d}\, x = -p(1)[u'(1)v(1) - u(1)v'(1)] + p(0) \underbrace{\left[-\frac{\alpha_1}{\alpha_2} u(0)v(0) + \frac{\alpha_1}{\alpha_2} u(0)v(0) \right]}_{0}$$

Now if we do the same for the β terms, we get

$$\int_0^1 (L[u]v - uL[v]) \, \mathrm{d}\, x = -p(1) \underbrace{\left[-\frac{\beta_1}{\beta_2} u(1)v(1) + \frac{\beta_1}{\beta_2} u(1)v(1) \right]}_{0} = 0$$

$$\int_0^1 (L[u]v - uL[v])\, \mathrm{d}\, x = 0 \implies \langle L[u], v\rangle - \langle u, L[v]\rangle = 0$$

Example 11.5 (Problem 22)**.**

$$\int_0^1 L[u]\overline{v}\, \mathrm{d}\, x = 0$$

Theorem 11.1. *All the eigenvalues of the Sturm-Liouville problem are real.*

Proof. Suppose $\lambda = a + b\imath$ is an eigenvalue of a Sturm-Liouville function, and suppose $\phi = U + V\imath$ is the corresponding eigenfunction. We want to show that $b = 0$, and that $V \equiv 0$. Set $u = \phi$ and $v = \phi$. This gives us

$$\int_0^1 (L[\phi]\overline{\phi} - \phi\overline{L[\phi]})\, \mathrm{d}\, x$$
$$\int_0^1 (\lambda r(x)\phi\overline{\phi} - \phi\overline{\lambda r(x)\phi})\, \mathrm{d}\, x$$

The conjugate of a real number is itself, so we can rewrite this as

$$\int_0^1 (\lambda r(x)\phi\overline{\phi} - \overline{\lambda} r(x)\phi\overline{\phi})\, \mathrm{d}\, x = 0$$
$$\int_0^1 (\lambda - \overline{\lambda}) r(x)\phi\overline{\phi}\, \mathrm{d}\, x = 0$$
$$(\lambda - \overline{\lambda}) \int_0^1 r(x)\phi\overline{\phi}\, \mathrm{d}\, x = 0$$
$$(\lambda - \overline{\lambda}) \int_0^1 r(x)(U^2 + V^2)\, \mathrm{d}\, x = 0$$

In order for this expression to be zero, either $r(x) \equiv 0$, which would have stopped us from ever reaching Laplace's identity, U and V could both be zero, but we would have a trivial case, or finally $lambda - \overline{\lambda} = 0$, which means $\lambda = \overline{\lambda}$, which means λ is real. □

Theorem 11.2 (Orthogonality property of eigenfunctions w.r.t. a weight function $r(x)$)**.** *If ϕ_m and ϕ_n are two eigenfunctions of the Sturm-Liouville problem, corresponding to eigenvalues λ_m and*

λ_n, *respectively, and if* $\lambda_m \neq \lambda_n$, *then*

$$\int_0^1 r(x)\phi_m\phi_n \, \mathrm{d}\, x = 0$$

Example 11.6 (Reexamining an earlier problem)**.** If we take the Sturm-Liouville problem with $p(x) \equiv 1$, $q(x) \equiv 0$, $r(x) \equiv 0$, $\alpha_1 = \beta_1 = 1$, and $\alpha_2 = \beta_2 = 0$, we get

$$y'' + \lambda y = 0; \quad y(0) = 0; \quad y(1) = 0$$

$$\lambda_n = (n\pi)^2; \quad \phi_n = \sin(n\pi x)$$

If $m \neq n$, then

$$\int_0^1 1 \cdot \sin(m\pi x)\sin(n\pi x) \, \mathrm{d}\, x = 0$$

We can start from the point

$$\langle L[u], v\rangle - \langle u, L[v]\rangle = 0.$$

We set $u = \phi_m$, and $v = \phi_n$.

$$\langle L[\phi_m], \phi_n\rangle - \langle \phi_m, L[\phi_n]\rangle = \int_0^1 (\lambda_m r(x)\phi_m\overline{\phi_n} - \phi_m\overline{\lambda_n r(x)\phi_n}) \, \mathrm{d}\, x$$

We know that all $r(x)$, λ, and ϕ are real, so we can rewrite this as

$$\int_0^1 (\lambda_m r(x)\phi_m\phi_n - \lambda_n r(x)\phi_m\phi_n) \, \mathrm{d}\, x = \int_0^1 (\lambda_m - \lambda_n) r(x)\phi_m\phi_n \, \mathrm{d}\, x = 0$$

We can pull out the λ terms, as they do not depend on x, leaving us with Theorem 11.2.

$$\int_0^1 r(x)\phi_n^2(x) \, \mathrm{d}\, x = 1; \quad n \in \mathbb{N}$$

Really, the integral will just give us some non-zero constant. But ϕ_n is an eigenfunction, therefore any constant multiple of it is still an eigenfunction, meaning we can choose a constant such that the result is 1.

Definition 11.2 (Kronecker's delta function)**.**

$$\delta_{m,n} = \int_0^1 r(x)\phi_m(x)\phi_n(x)\ \mathrm{d}\,x = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{11.3}$$

ϕ_n is said to be orthornormal, meaning it is both orthoganal and has unit length.

Example 11.7 (Determine the normalized eigenfunction)**.**

$$y'' + \lambda y = 0; \quad y(0) = 0; \quad y(1) = 0$$
$$\lambda_n = (n\pi)^2; \quad \phi_n = \sin(n\pi x)$$

So we know that $r(x) \equiv 1$, so we have

$$\phi_n(x) = c_n \sin(n\pi x)$$
$$\delta_{m,n} = c_n^2 \int_0^1 \sin^2(n\pi x)\ \mathrm{d}\,x = 1$$
$$\delta_{m,n} = c_n^2 \cdot \frac{1}{2} = 1 \implies c_n = \sqrt{2}$$
$$\phi_n(x) = \sqrt{2}\sin(n\pi x)$$

Example 11.8 (Determine the normalized eigenfunction)**.**

$$y'' + \lambda y = 0; \quad y(0) = 0; \quad y'(1) + y(1) = 0$$
$$\sin(\sqrt{\lambda_n}) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) = 0$$
$$\phi_n(x) = c_n \sin(\sqrt{\lambda_n}x)$$

Again, $r(x) \equiv 1$, so

$$\delta_{m,n} = c_n^2 \int_0^1 \sin^2(\sqrt{\lambda_n}x)\ \mathrm{d}\,x = 1$$
$$\delta_{m,n} = c_n^2 \left[\frac{1}{2} - \frac{\sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}}\right] = 1$$

$$c_n^2\left[\frac{\sqrt{\lambda_n}-\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{2\sqrt{\lambda_n}}\right]=1$$

Now we can solve the given eigenvalue expression for $\sin(\sqrt{\lambda_n})$, and plug it in here

$$\begin{aligned}
c_n^2\left[\frac{\sqrt{\lambda_n}+\sqrt{\lambda_n}\cos^2(\sqrt{\lambda_n})}{2\sqrt{\lambda_n}}\right]&=1\\
c_n^2\left[\frac{1+\cos^2(\sqrt{\lambda_n})}{2}\right]&=1\\
c_n=\sqrt{\frac{2}{1+\cos^2(\sqrt{\lambda_n})}}&=1\\
\phi_n(x)=\sqrt{\frac{2}{1+\cos^2(\sqrt{\lambda_n})}}&\sin(\sqrt{\lambda_n}x)
\end{aligned}$$

We can approximate any function $f(x)$ with any eigenfunction $\phi_n(x)$ which satisfies $\int_0^1 r(x)\phi_m(x)\phi_n(x)\,\mathrm{d}\,x=\delta_{m,n}$, using

$$f(x)=\sum_{n=1}^{\infty}c_n\phi_n(x). \tag{11.4}$$

Now we'll find c_n

$$\begin{aligned}
f(x)&=\sum_{n=1}^{\infty}c_n\phi_n(x)\\
r(x)\phi_m(x)f(x)&=\sum_{n=1}^{\infty}c_n r(x)\phi_m(x)\phi_n(x)\\
\int_0^1 r(x)\phi_m(x)f(x)\,\mathrm{d}\,x&=\int_0^1\sum_{n=1}^{\infty}c_n r(x)\phi_m(x)\phi_n(x)\,\mathrm{d}\,x\\
\int_0^1 r(x)\phi_m(x)f(x)\,\mathrm{d}\,x&=\sum_{n=1}^{\infty}c_n\int_0^1 r(x)\phi_m(x)\phi_n(x)\,\mathrm{d}\,x\\
\int_0^1 r(x)\phi_m(x)f(x)\,\mathrm{d}\,x&=\sum_{n=1}^{\infty}c_n\delta_{m,n}
\end{aligned}$$

But when $m\neq n$, $\delta_{m,n}=0$, so the sum is zero everywhere except when $m=n$, simplifying the equation to

$$c_m=\int_0^1 r(x)\phi_m(x)f(x)\,\mathrm{d}\,x$$

So the entire relationship is

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x); \quad c_n = \int_0^1 r(x)\phi_n(x)f(x) \, \mathrm{d}\, x \tag{11.5}$$

Example 11.9.

$$f(x) = x, \quad 0 \leq x \leq 1.$$

Write $f(x)$ in terms of $\phi_n(x) = k_n \sin(\sqrt{\lambda_n} x)$. We found previously that

$$k_n = \sqrt{\frac{2}{1 + \cos^2(\sqrt{\lambda_n})}}$$

We know that

$$f(x) = \int_{n=1}^{\infty} c_n \phi_n(x); \quad c_n = \int_0^1 r(x)\phi_n(x)f(x) \, \mathrm{d}\, x.$$

Since we know $\phi_n(x)$, we just need to solve for c_n

$$c_n = \int_0^1 k_n \sin(\sqrt{\lambda_n} x) \cdot x \, \mathrm{d}\, x$$

$$c_n = k_n \left[\frac{\sin(\sqrt{\lambda_n})}{\lambda_n} - \frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right] = \sqrt{\frac{2}{1 + \cos^2(\sqrt{\lambda_n})}} \left[\frac{\sin(\sqrt{\lambda_n})}{\lambda_n} - \frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right]$$

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2}{1 + \cos^2(\sqrt{\lambda_n})} \left[\frac{\sin(\sqrt{\lambda_n})}{\lambda_n} - \frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right] \sin(\sqrt{\lambda_n} x)$$

11.3 Nonhomogenous Boundary Value Problems

For homogeneous BVP's, we had

$$L[y] = \lambda r(x) y \tag{11.6}$$

with boundary conditions

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases} \tag{11.7}$$

For a nonhomogeneous BVP, we instead have

$$L[y] = \mu r(x) y + f(x), \tag{11.8}$$

while the boundary conditions remain the same.

Let $\phi(x)$ be a solution to BVP (11.8) with boundary conditions (11.7).

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x), \tag{11.9}$$

where ϕ_n are the orthonormal eigenfunctions for (11.6). i.e.

$$L[\phi_n] = \lambda_n r(x) \phi_n(x).$$

$$b_n = \int_0^1 r(x)\phi(x)\phi_n(x) \, \mathrm{d}\, x \tag{11.10}$$

$$\begin{aligned}
L[\phi] &= \mu r(x)\phi(x) + f(x) \\
L\left[\sum_{n=1}^{\infty} b_n \phi_n(x)\right] &= \mu r(x)\phi(x) + f(x) \\
\sum_{n=1}^{\infty} b_n L[\phi_n(x)] &= \mu r(x)\phi(x) + f(x) \\
\sum_{n=1}^{\infty} b_n \lambda_n r(x)\phi_n(x) &= \mu r(x) \sum_{n=1}^{\infty} b_n \phi_n(x) + r(x)\frac{f(x)}{r(x)} \\
\frac{f(x)}{r(x)} = g(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\
c_n &= \int_0^1 r(x)\frac{f(x)}{r(x)}\phi_n(x) \, \mathrm{d}\, x \\
\sum_{n=1}^{\infty} b_n \lambda_n r(x)\phi_n(x) &= \sum_{n=1}^{\infty} \mu r(x) b_n \phi_n(x) - \sum_{n=1}^{\infty} r(x) c_n \phi_n(x)
\end{aligned}$$

$$\sum_{n=1}^{\infty} \left[b_n \lambda_n - \mu b_n - c_n\right] r(x)\phi_n(x) = 0$$

$$r(x)\sum_{n=1}^{\infty}\left[b_n\lambda_n - \mu b_n - c_n\right]\phi_n(x) = 0$$

$r(x)$ is non-zero, so we can divide both sides by it to eliminate it, leaving us with

$$\sum_{n=1}^{\infty}\left[(\lambda_n - \mu)b_n - c_n\right]\phi_n(x) = 0$$

$$(\lambda_n - \mu)b_n - c_n = 0$$

$$\text{Case 1: } \lambda_n \neq \mu \ \forall \ n$$

$$b_n = \frac{c_n}{\lambda_n - \mu}; \quad c_n = \int_0^1 f(x)\phi_n(x)\,\mathrm{d}\,x$$

$$\phi(x) = \sum_{n=1}^{\infty}\left(\frac{c_n}{\lambda_n - \mu}\right)\phi_n(x)$$

$$\text{Case 2: } \exists\ m \ni \lambda_m = \mu$$

$$0 \cdot b_m - c_m = 0 \implies 0 - c_m = 0$$

$$\text{Subcase 1: } c_m = 0 \implies b_m \text{ can be arbitrary} \implies \text{infinitely many solutions to } \phi(x)$$

$$\text{Subcase 2: } c_m \neq 0 \implies \text{ No solution}$$

In Subcase 1: $c_m = 0$ implies that $c_m = \int_0^1 f(x)\phi_m(x)\,\mathrm{d}\,x = 0$, so $f(x)$ is orthogonal to $\phi_m(x)$.

Fredholm's Alternative.

Example 11.10 (Solve the BVP)**.**

$$y'' + 2y = -x$$

$$y(0) = 0; \quad y(1) + y'(1) = 0$$

Look at the Sturm–Liouville problem

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x) = \lambda r(x)y$$

The homogenous solution to this problem is

$$y'' + \lambda y = 0$$
$$\begin{cases} y(0) = 0 \\ y(1) + y'(1) = 0 \end{cases}$$
$$\phi_n(x) = k_n \sin(\sqrt{\lambda_n} x)$$
$$k_n = \sqrt{\frac{2}{1 + \cos^2(\sqrt{\lambda_n})}}$$
$$\sin(\sqrt{\lambda_n}) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) = 0$$

From this we find $\lambda_1 \approx 4.116$. Our μ is the 2 from the original problem. Since $\lambda_1 > \mu$, and $\lambda_{n+1} > \lambda_n \ \forall n$, this means that $\lambda_n > \mu \ \forall n$, and therefore they are never equal, making this Case 1. Meaning

$$b_n = \frac{c_n}{\lambda_n - \mu},$$

where

$$c_n = \int_0^1 f(x)\phi_n(x) \, \mathrm{d}\, x$$

Here $f(x) = x$, so

$$c_n = k_n \int_0^1 x \sin(\sqrt{\lambda_n} x) \, \mathrm{d}\, x = k_n \left[\frac{\sin(\sqrt{\lambda_n})}{\lambda_n} - \frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right]$$
$$\phi(x) = \sum_{n=1}^{\infty} \frac{k_n^2}{\lambda_n - 2} \left[\frac{\sin(\sqrt{\lambda_n})}{\lambda_n} - \frac{\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}} \right] \sin(\sqrt{\lambda_n} x)$$

Non-homogeneous heat conduction problem

$$r(x)U_t = [p(x)U_x]_x - q(x)U + F(x,t) \tag{11.11}$$

with boundary conditions

$$\begin{cases} U_x(0,t) - h_1 U(0,t) = 0 \\ U_x(1,t) + h_2 U(1,t) = 0 \end{cases} \tag{11.12}$$

with initial condition

$$U(x,0) = f(x). \tag{11.13}$$

When we set $F(x,t) = 0$,

$$-[p(x)X']' + q(x)X = \lambda r(x)X$$

$$X'(0) - h_1 X(0) = 0; \quad X'(1) + h_2 X(1) = 0$$

$$U(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

$$U_x(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n'(x)$$

$$b_n(t) = \int_0^1 r(x)U(x,t)\phi_n(x)\,\mathrm{d}\,x$$

$$[p(x)U_x]_x - q(x)U = \left[p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n'(x)\right]_x - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

$$r(x)U_t = r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x)$$

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \left[p(x)\sum_{n=1}^{\infty} b_n(t)\phi_n'(x)\right]_x - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\,[p(x)\phi_n'(x)]' - q(x)\sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\left[[p(x)\phi_n'(x)]' - q(x)\phi_n(x)\right]$$

Recall that $L[u] = \lambda r(x)u = -[p(x)u']' + q(x)u$, so

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\left[-\lambda_n r(x)\phi_n(x)\right] = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x)$$

$$[p(x)U_x]_x - q(x)U = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x)$$

$$\text{Similarly: } r(x)U_t = r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x)$$

Now that we have everything for the homogeneous case, we can use those values in the nonhomogeneous case

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) + F(x,t)$$

$$r(x)\frac{F(x,t)}{r(x)} = r(x)\sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x), \text{ where } \gamma_n(t) = \int_0^1 r(x)\frac{F(x,t)}{r(x)}\phi_n(x)\ \mathrm{d}\,x$$

$$r(x)\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = -r(x)\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) + r(x)\sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

$$\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = -\sum_{n=1}^{\infty} b_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty}\gamma_n(t)\phi_n(x)$$

$$\sum_{n=1}^{\infty}\left[b_n'(t) + b_n(t)\lambda_n - \gamma_n(t)\right]\phi_n(x) = 0$$

$$\phi_n(x) \neq 0 \implies b_n'(t) + b_n(t)\lambda_n - \gamma_n(t) = 0$$

Doesn't this look like $y'(t) + p(t)y(t) = q(t)$, from Chapter 2? We can use integrating factor, where $\mu(t) = \exp\left(\int p(t)\ \mathrm{d}\,t\right)$

$$\mu(t) = \exp\left(\int \lambda_n\,\mathrm{d}\,t\right) = \exp\left(\lambda_n t\right)$$

$$\exp\left(\lambda_n t\right)b_n'(t) + \exp\left(\lambda_n t\right)b_n(t)\lambda_n = \exp\left(\lambda_n t\right)\gamma_n(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[\exp\left(\lambda_n t\right)b_n(t)\right] = \exp\left(\lambda_n t\right)\gamma_n(t)$$

$$\int \frac{\mathrm{d}}{\mathrm{d}\,t}\left[\exp\left(\lambda_n t\right)b_n(t)\right]\mathrm{d}\,t = \int \exp\left(\lambda_n t\right)\gamma_n(t)\,\mathrm{d}\,t$$

$$\exp\left(\lambda_n t\right)b_n(t) = \int \exp\left(\lambda_n t\right)\gamma_n(t)\,\mathrm{d}\,t = \int_0^t \exp\left(\lambda_n s\right)\gamma_n(s)\,\mathrm{d}\,s + c$$

$$b_n(t) = \int_0^t \exp(\lambda_n(s-t))\gamma_n(s)\ \mathrm{d}\,s + c\exp(-\lambda_n t)$$

$$U(x,0) = \sum_{n=1}^{\infty} b_n(0)\phi_n(x) = f(x)$$

$$b_n(0) = \int_0^0 \ldots\mathrm{d}\,s + c\exp(0) = c$$

11.4 Singular Sturm–Liouville Problems

In the preceding sections of this chapter, we considered Sturm–Liouville boundary value problems: the differential equation

$$L[y] = -[p(x)y']' + q(x)y = \lambda r(x)y, \quad 0 < x < 1 \tag{11.14}$$

together with boundary conditions of the form

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases} \tag{11.15}$$

Then, we assumed that $r(x), p(x) > 0$, but that is not always the case.

One example is Bessel's equation, of order v, on the interval $0 < x < 1$. This equation can be written in the form

$$-(xy')' + \frac{v^2}{x}y = \lambda xy, \tag{11.16}$$

meaning $p(x) = x$, $q(x) = v^2/x$, and $r(x) = x$. Now there are a few things to note about $q(x)$

$$\begin{cases} \lim_{x\to 0^+} q(x) = +\infty \\ \lim_{x\to 0^-} q(x) = -\infty \\ \lim_{x\to 0} q(x) = \text{undef.} \end{cases}$$

Another example is Legendre's equation

$$-[(1-x^2)y']' = \lambda y, \quad -1 < x < 1, \tag{11.17}$$

where $\lambda = \alpha(\alpha+1)$, $p(x) = 1 - x^2$, and $q(x) = 0$. The conditions are satisfied on p, q, and r on the interval $-1 \leq x \leq 1$, except at $x = \pm 1$, where $p(x) = 0$.

Another example is

$$xy'' + y' + \lambda xy; \quad 0 < x < 1; \quad \lambda > 0, \tag{11.18}$$

which can be rewritten as

$$-\left[xy'\right]' = \lambda xy. \tag{11.19}$$

Now we can see that $p(x) = r(x) = x$, and $q(x) = 0$.

Now we introduce a new variable $t := \sqrt{\lambda}x$.

$$\begin{gathered}
\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = y' = \frac{\mathrm{d}\,y}{\mathrm{d}\,t}\frac{\mathrm{d}\,t}{\mathrm{d}\,x} \\
\frac{\mathrm{d}\,t}{\mathrm{d}\,x} = \sqrt{\lambda} \implies \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \sqrt{\lambda}\frac{\mathrm{d}\,y}{\mathrm{d}\,t} \\
\frac{\mathrm{d}^2\,y}{\mathrm{d}\,x} = y'' = \frac{\mathrm{d}}{\mathrm{d}\,x}\left[\sqrt{\lambda}\frac{\mathrm{d}\,y}{\mathrm{d}\,t}\right] = \frac{\mathrm{d}}{\mathrm{d}\,t}\left[\sqrt{\lambda}\frac{\mathrm{d}\,y}{\mathrm{d}\,t}\right]\frac{\mathrm{d}\,t}{\mathrm{d}\,x} = \lambda\frac{\mathrm{d}^2\,y}{\mathrm{d}\,t^2} \\
\text{In general: } \frac{\mathrm{d}^n\,y}{\mathrm{d}\,x^n} = y^{(n)} = \lambda^{n/2}\frac{\mathrm{d}^n\,y}{\mathrm{d}\,t^n}
\end{gathered}$$

Now we can rewrite the problem as

$$\frac{t}{\sqrt{\lambda}}\lambda y''(t) + \sqrt{\lambda}y'(t) + \lambda\frac{t}{\sqrt{\lambda}}y = 0.$$

We know that $\sqrt{\lambda} > 0$, therefore we can divide both sides by $\sqrt{\lambda}$, which comes out to be

$$ty''(t) + y'(t) + ty(t) = 0 \tag{11.20}$$

Equation 11.20 is Bessel's equation of order zero (see Section 5.7). The general solution for $t > 0$ is

$$y = c_1 J_0(\sqrt{\lambda}x) + c_2 Y_0(\sqrt{\lambda}x), \tag{11.21}$$

where J_0 and Y_0 denote the Bessel functions of the first and second kinds of order zero.

$$J_0(\sqrt{\lambda}x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^m x^{2m}}{2^{2m}(m!)^2}, \quad x > 0, \tag{11.22}$$

$$Y_0(\sqrt{\lambda}x) = \frac{2}{\pi}\left[\left(\gamma + \ln\left[\frac{\sqrt{\lambda}x}{2}\right]\right) J_0(\sqrt{\lambda}x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m \lambda^m x^{2m}}{2^{2m}(m!)^2}\right], \quad x > 0, \tag{11.23}$$

$$H_m = \sum_{n=1}^{m} \frac{1}{n}, \tag{11.24}$$

$$\gamma = \lim_{m\to\infty}(H_m - \ln m). \tag{11.25}$$

For Equations 11.18 and 11.19 with the boundary conditions $y(0) = y(1) = 0$.

$$J_0(0) = 1; \quad \lim_{t\to 0} Y_0(t) = -\infty \implies c_2 = 0,$$

so

$$y(x) = c_1 J_0(\sqrt{\lambda}x).$$

We also have the boundary condition $y(1) = 0$, so

$$y(1) = 0 = c_1 J_0(\sqrt{\lambda}),$$

and for this to be non-trivial, $c_1 \neq 0$, meaning

$$J_0(\sqrt{\lambda}) = 0. \tag{11.26}$$

If, however, we consider the other boundary condition, $y(0) = 0$,

$$y(0) = 0 = c_1 J_0(0) = c_1 \implies c_1 = c_2 = 0,$$

so the only solution is the trivial solution.

Let's modify the boundary condition, such that y and y' are bounded as x approaches 0. We can still keep the other boundary condition that $y(1) = 0$. We are going to do everything on the open interval $[\epsilon, 1]$, and in the very end we will take the limit as $\epsilon \to 0$.

Now we look at (11.26).

$$J_0(\sqrt{\lambda}) = 0 = \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^m}{4^m (m!)^2} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \lambda^m}{4^m (m!)^2}$$

We'll need a CAS to find the eigenfunctions and eigenvalues of this function

$$\phi_n(x) = y_n(x) = J_0(\sqrt{\lambda_n} x)$$
$$\int_0^1 r(x)\phi_m(x)\phi_n(x)\ \mathrm{d}\,x = 0 \mid m \neq n$$
$$\int_0^1 x J_0(\sqrt{\lambda_m} x) J_0(\sqrt{\lambda_n} x)\ \mathrm{d}\,x = 0$$

For a given $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Multiply both sides by $r(x)\phi_m(x)$ and integrate from 0 to 1

$$r(x)\phi_m(x)f(x) = \sum_{n=1}^{\infty} c_n \phi_m(x) f(x) \phi_n(x)$$
$$\int_0^1 r(x)\phi_m(x)f(x)\ \mathrm{d}\,x = \int_0^1 \sum_{n=1}^{\infty} c_n \phi_m(x) r(x) \phi_n(x)\ \mathrm{d}\,x$$
$$\int_0^1 r(x)\phi_m(x)f(x)\ \mathrm{d}\,x = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_m(x) r(x) \phi_n(x)\ \mathrm{d}\,x$$
$$\int_0^1 r(x)\phi_m(x)f(x)\ \mathrm{d}\,x = c_m \int_0^1 r(x)\phi_m^2(x)\ \mathrm{d}\,x$$
$$c_m = \frac{\int_0^1 r(x)\phi_m(x)f(x)\ \mathrm{d}\,x}{\int_0^1 r(x)\phi_m^2(x)\ \mathrm{d}\,x}$$

If $\phi_m(x)$ were normalized, then c_m would simply be

$$c_m = \int_0^1 r(x)\phi_m(x)f(x)\ \mathrm{d}\,x$$

We just did the homogeneous case. What about the non-homogeneous case?

$$\begin{cases} -[xy']' = \mu xy + f(x) \\ y,\ y' \text{ are bounded as } x \to 0, \\ y(1) = 0 \end{cases}$$

Note that μ is not an eigenvalue of the homogeneous case.

We begin by finding the solution to the homogeneous case, where $f(x) = 0$. This solution, as found earlier, is

$$\phi_n(x) = J_0(\sqrt{\lambda_n} x)$$

Now we look for the b_n that satisfies

$$\phi(x) = \sum_{n=1}^{\infty} b_n \phi_n(x),$$

which means it also satisfies

$$(\lambda_n - \mu) b_n - c_n = 0.$$

Assuming that $\lambda_n \neq \mu$, b_n is given by

$$b_n = \frac{c_n}{\lambda_n - \mu},$$

so y is given by

$$y(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \phi_n(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n} x),$$

where $J_0(\sqrt{\lambda_n}) = 0$. Now c_n is given by

$$c_n = \frac{\int_0^1 f(x)\phi_m(x)\ \mathrm{d}\,x}{\int_0^1 r(x)\phi_m^2(x)\ \mathrm{d}\,x} = \frac{\int_0^1 f(x) J_0(\sqrt{\lambda_n} x)\ \mathrm{d}\,x}{\int_0^1 x J_0^2(\sqrt{\lambda_n} x)\ \mathrm{d}\,x}$$

Back in 11.2, we solved Sturm–Liouville problems in the following way

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0 \tag{11.27}$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0 \tag{11.28}$$

$$\beta_1 y(1) + \beta_2 y'(1) = 0 \tag{11.29}$$

$$\int_0^1 (L[u]v - uL[v])\,\mathrm{d}\,x = -p(x)[u'(x)v(x) - u(x)v'(x)]_0^1 = 0. \tag{11.30}$$

However, now the function is singular at $x = 0$, so what we do instead is look at a point very close to $x = 0$, which we will call ϵ, and then take the limit as $\epsilon \to 0$.

$$\begin{aligned}\int_\epsilon^1 (L[u]v - uL[v])\,\mathrm{d}\,x &= -p(x)[u'(x)v(x) - u(x)v'(x)]_\epsilon^1 \\ &= -p(1)[u'(1)v(1) - u(1)v'(1)] + p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] \\ &= 0 + p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] \\ \int_0^1 (L[u]v - uL[v])\,\mathrm{d}\,x &= \lim_{\epsilon\to 0} p(\epsilon)[u'(\epsilon)v(\epsilon) - u(\epsilon)v'(\epsilon)] = 0\end{aligned}$$

If, instead, we had a singularity at $x = 1$, then instead we do

$$\begin{aligned}\int_0^{1-\epsilon} (L[u]v - uL[v])\,\mathrm{d}\,x &= -p(x)[u'(x)v(x) - u(x)v'(x)]_0^{1-\epsilon} \\ &= -p(1-\epsilon)[u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)] + p(0)[u'(0)v(0) - u(0)v'(0)] \\ &= -p(1-\epsilon)[u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)] \\ \int_0^1 (L[u]v - uL[v])\,\mathrm{d}\,x &= \lim_{\epsilon\to 0} -p(1-\epsilon)[u'(1-\epsilon)v(1-\epsilon) - u(1-\epsilon)v'(1-\epsilon)] = 0\end{aligned}$$

Example 11.11.

$$\begin{cases} a^2\left[U_{rr} + \frac{1}{r}U_r\right] = U_{tt} \\ 0 \le r \le 1, \quad t \ge 0 \\ u(1,t) = 0, \quad t \ge 0 \\ u(r,0) = f(r); \quad u_t(r,0) = 0 \end{cases}$$

$$u(r,t) = R(r)T(t)$$

$$a^2\left[R''T + \frac{1}{r}R'T\right] = RT''$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{T''}{a^2T} := -\lambda^2, \quad \lambda > 0$$

$$\begin{cases} r^2R'' + rR' + r^2\lambda^2R = 0 \\ T'' + a^2\lambda^2T = 0 \end{cases}$$

$$x := \lambda r \implies \frac{\mathrm{d}\,x}{\mathrm{d}\,r} = \lambda$$

$$\frac{\mathrm{d}\,R}{\mathrm{d}\,r} = \frac{\mathrm{d}\,R}{\mathrm{d}\,x}\frac{\mathrm{d}\,x}{\mathrm{d}\,r} = \frac{\mathrm{d}\,R}{\mathrm{d}\,x}\lambda \implies \frac{\mathrm{d}^2\,R}{\mathrm{d}\,r^2} = \frac{\mathrm{d}^2\,R}{\mathrm{d}\,x^2}\lambda^2$$

$$\frac{x^2}{\lambda^2}\left[\lambda^2\frac{\mathrm{d}^2\,R}{\mathrm{d}\,x^2}\right] + \frac{x}{\lambda}\left[\lambda\frac{\mathrm{d}\,R}{\mathrm{d}\,x}\right] + x^2R = 0$$

$$x^2\frac{\mathrm{d}^2\,R}{\mathrm{d}\,x^2} + x\frac{\mathrm{d}\,R}{\mathrm{d}\,x} + x^2R = 0$$

This is Bessel's equation of order zero, so the solution is

$$R(x) = c_1J_0(x) + c_2Y_0(x) \implies R(r) = c_1J_0(\lambda r) + c_2Y_0(\lambda r)$$

Simply using the characteristic equation, we can find $T(t)$

$$T(t) = k_1\sin(\lambda at) + k_2\cos(\lambda at)$$

Now we use our initial and boundary conditions

$$u(1,t) = R(1)T(t) = 0 \implies R(1) = 0$$

$$R(r) = c_1J_0(\lambda r)$$

$$R(1) = 0 \implies c_1J_0(\lambda) = 0 \implies J_0(\lambda) = 0$$

$$u(r,t) = \sum_{n=1}^{\infty} J_0(\lambda_n r)\left[c_n\sin(\lambda_n at) + k_n\cos(\lambda_n at)\right]$$

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} J_0(\lambda_n r)\left[c_n\sin(0) + k_n\cos(0)\right] = \sum_{n=1}^{\infty} k_nJ_0(\lambda_n r)$$

$$k_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\ \mathrm{d}\,r}{\int_0^1 r J_0^2(\lambda_n r)\ \mathrm{d}\,r}$$

$$u_t(r,0) = \sum_{n=1}^{\infty} c_n \lambda_n a J_n(\lambda_n r)\cos(0) = \sum_{n=1}^{\infty} c_n \lambda_n a J_n(\lambda_n r) = 0 \implies c_n = 0$$

$$\begin{cases} u(r,t) = \sum_{n=1}^{\infty} k_n J_0(\lambda_n r)\cos(\lambda_n a t) \\ k_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\ \mathrm{d}\,r}{\int_0^1 r J_0^2(\lambda_n r)\ \mathrm{d}\,r} \end{cases}$$

11.6 Series of Orthogonal Functions: Mean Convergence

Suppose we have orthonormal eigenfunctions $\phi_n(x)$ which satisfy

$$\int_0^1 r(x)\phi_m(x)\phi_n(x)\ \mathrm{d}\,x = \delta_{mn}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

The n^{th} partial sum is

$$S_n(x) = \sum_{i=1}^{n} a_i \phi_i(x). \tag{11.31}$$

We are looking for the coefficients a_i such that S_n "best approximates" $f(x)$ on the interval $0 \leq x \leq 1$. There are several ways to do this.

The first method we consider is called the **collocation method**. We choose n values of x, labelled x_j, such that

$$f(x_j) = \sum_{i=1}^{n} a_i \phi_i(x_j). \tag{11.32}$$

This gives us a system of equations.

Alternatively, we can consider the difference

$$|f(x) - S_n(x)| \tag{11.33}$$

and try to make it as small as possible. The problem here is that $S_n(x)$ also depends on the coefficients a_i. One way to proceed is to instead consider the least upper bound of the difference (lub).

$$E_n(a_1, \ldots, a_n) = \text{lub}_{0 \leq x \leq 1} |f(x) - S_n(x)| \tag{11.34}$$

Another way is to consider

$$I_n(a_1, \ldots, a_n) = \int_0^1 r(x)|f(x) - S_n(x)| \, \mathrm{d}\, x, \tag{11.35}$$

If $r(x) = 1$, then I_n is the area between the plots of the two functions. Since absolute values make calculations more complicated, we can instead use the mean square error (MSE), R_n, given by

$$R_n(a_1, \ldots, a_n) = \int_0^1 r(x) \left[f(x) - S_n(x)\right]^2 \, \mathrm{d}\, x. \tag{11.36}$$

To minimize this, we take the partial derivatives

$$\frac{\partial R_n}{\partial a_i} = 0, \tag{11.37}$$

which we compute as follows:

$$\begin{aligned} R_n(x) &= \int_0^1 r(x) \left[f(x) - \sum_{i=1}^{n} a_i \phi_i(x)\right]^2 \, \mathrm{d}\, x \\ \frac{\partial R_n}{\partial a_i} &= \int_0^1 r(x) 2 \left[f(x) - \sum_{j=1}^{n} a_j \phi_j\right] \phi_i \, \mathrm{d}\, x = 0 \end{aligned}$$

we can divide by 2 throughout, and distribute $r(x)$ and $\phi_i(x)$.

$$\begin{aligned} 0 &= \int_0^1 r(x) f(x) \phi_i(x) \, \mathrm{d}\, x - \int_0^1 r(x) \sum_{j=1}^{n} a_j \phi_j \phi_i \, \mathrm{d}\, x \\ 0 &= \int_0^1 r(x) f(x) \phi_i(x) \, \mathrm{d}\, x - \sum_{j=1}^{n} a_j \int_0^1 r(x) \phi_j \phi_i \, \mathrm{d}\, x \end{aligned}$$

Since $\phi_i(x)$ is orthornormal, that means that the integral on the right is zero when $j \neq i$, and one when $j = i$, so

$$0 = \int_0^1 r(x)f(x)\phi_i(x) \, \mathrm{d}\, x - a_i$$
$$a_i = \int_0^1 r(x)f(x)\phi_i(x) \, \mathrm{d}\, x$$

Inequalities

Cauchy's Inequality

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n \leq \sqrt{a_1^2 + \ldots + a_n^2}\sqrt{b_1^2 + \ldots + b_n^2}$$

Proof by induction.

Base case: show it is true for $n = 1$.

$$a_1b_1 \leq \sqrt{a_1^2}\sqrt{b_1^2}$$
$$a_1b_1 \leq a_1b_1$$
$$a_1b_1 = a_1b_1$$

This is trivial, so we are going to do it for $n = 2$ instead.

$$\text{W.T.S. } a_1b_1 + a_2b_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$$
$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$
$$a_1^2b_1^2 + 2a_1a_2b_1b_2 + a_2^2b_2^2 \leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$
$$2a_1a_2b_1b_2 \leq a_1^2b_2^2 + a_2^2b_1^2$$
$$2(a_1b_2)(a_2b_1) \leq (a_1b_2)^2 + (a_2b_1)^2$$
$$0 \leq (a_1b_2)^2 + (a_2b_1)^2 - 2(a_1b_2)(a_2b_1)$$
$$0 \leq (a_1b_2 - a_2b_1)^2 \text{ we know this is true}$$

Hypothesis case: assume it is true for $n = k$ and show that it is true for $n = k + 1$.

$$\text{assume that } \sum_{i=1}^{k} a_i b_i \le \sqrt{\sum_{i=1}^{k} a_i^2}\sqrt{\sum_{i=1}^{k} b_i^2}$$

$$\text{W.T.S. it is true for } n = k+1 \implies \sum_{i=1}^{k+1} a_i b_i \le \sqrt{\sum_{i=1}^{k+1} a_i^2}\sqrt{\sum_{i=1}^{k+1} b_i^2}$$

$$\sum_{i=1}^{k} a_i b_i + a_{k+1}b_{k+1} \le \underbrace{\sqrt{\sum_{i=1}^{k} a_i^2}}_{a_1}\underbrace{\sqrt{\sum_{i=1}^{k} b_i^2}}_{b_1} + \underbrace{a_{k+1}}_{a_2}\underbrace{b_{k+1}}_{b_2} \le \sqrt{\sum_{i=1}^{k} a_i^2 + a_{k+1}^2}\sqrt{\sum_{i=1}^{k} b_i^2 + b_{k+1}^2}$$

Example. Show that for each real sequence $a_1, a_2, \ldots, a_n$, one has

$$a_1 + a_2 + \ldots + a_n \le \sqrt{n}\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

This is just Cauchy's inequality where $b_1 = b_2 = \ldots = b_n = 1$. The square root term with b_i will come out to be $\sqrt{n}$, and that's it.

Example. Show that one has

$$\sum_{i=1}^{n} a_i \le \sqrt{\sum_{i=1}^{n} |a_i|^{2/3}}\sqrt{\sum_{i=1}^{n} |a_i|^{4/3}}$$

This is the same as

$$\sum_{i=1}^{n} a_i \le \sqrt{\sum_{i=1}^{n} (|a_i|^{1/3})^2}\sqrt{\sum_{i=1}^{n} (|a_i|^{2/3})^2}$$

This is just Cauchy's inequality with $a_i = |a_i|^{1/3}$ and $b_i = |b_i|^{1/3}$. Note that when you multiply these two together, the 1/3 and 2/3 powers combine to just 1.

Example.

$$f(x) := \cos(\beta x) \implies f^2(x) = \frac{1 + f(2x)}{2}$$

Show that if $P_i \geq 0$, for $1 \leq i \leq n$, and $\sum_{i=1}^n P_i = 1$, then

$$g(x) = \sum_{i=1}^{n} P_i \cos(\beta_i x)$$

satisfies

$$g^2(x) \leq \frac{1 + g(2x)}{2}$$

If we fix x, then we can set $a_i := P_i$ and $b_i := \cos(\beta_i x)$, which would mean

$$g(x) = \sum_{i=1}^{n} P_i \cos(\beta_i x) \leq \sqrt{\sum_{i=1}^{n} P_i^2} \sqrt{\sum_{i=1}^{n} \cos(\beta_i x)},$$

but we don't know the sum of P_i^2, we know the sum of P_i. So instead we must set $a_i := \sqrt{P_i}$. If we only changed that, then the sum would be

$$\sum_{i=1}^{n} \sqrt{P_i} \cos(\beta_i x) \leq,$$

but we don't know that sum, because of the $\sqrt{P_i}$, so we must also set $b_i := \sqrt{P_i} \cos(\beta_i x)$, which will make the sum

$$g(x) = \sum_{i=1}^{n} P_i \cos(\beta_i x) \leq \sqrt{\sum_{i=1}^{n} P_i} \sqrt{\sum_{i=1}^{n} P_i \cos^2(\beta_i x)} = \sqrt{\sum_{i=1}^{n} P_i \cos^2(\beta_i x)}$$

Now we can square both sides

$$g^2(x) \leq \sum_{i=1}^{n} P_i \cos^2(\beta_i x) = \sum_{i=1}^{n} \frac{P_i [1 + \cos(2\beta_i x)]}{2} = \frac{1}{2} \sum_{i=1}^{n} [P_i + P_i \cos(2\beta_i x)]$$
$$g^2(x) \leq \frac{1}{2} \left[1 + \sum_{i=1}^{n} P_i \cos(2\beta_i x) \right] = \frac{1}{2} [1 + g(2x)]$$

Definition (Convex)**.** A function $f : [a, b] \to \mathbb{R}$ is said to be *convex* provided that

$$\forall\ x \in [a, b]\ \&\ \forall\ p \in [0, 1]$$

one has

$$f(px + (1-p)y) \leq pf(x) + (1-p)f(y).$$

A geometrical way of putting it is that if you draw a line segment between any two points on $f : [a, b]$, the line segment will be above the curve, so long as f is convex on the interval $[a, b]$.

Definition (Jensen's Inequality)**.** Suppose $f : [a, b] \to \mathbb{R}$ is a convex function and for $P_j \geq 0$ & $\sum_{j=1}^{n} P_j = 1$, then $\forall\, x_j \in [a, b],\ j = 1, \ldots, n$, one has

$$f\left(\sum_{j=1}^{n} P_j x_j\right) \leq \sum_{j=1}^{n} P_j f(x_j)$$

Proof. We will use proof by induction.

The base case is $n = 1$, which means $P_1 = 1$, and

$$f(P_1 x_1) \leq P_1 f(x_1) \implies f(1 \cdot x_1) \leq 1 \cdot f(x_1) \implies f(x_1) \leq f(x_1),$$

which is trivial, because they are equal. So we will instead use $n = 2$ as our base case.

We want to show that

$$f(P_1 x_1 + P_2 x_2) \leq P_1 f(x_1) + P_2 f(x_2); \quad P_1 + P_2 = 1 \implies P_2 = 1 - P_1$$

By convexity, we know the following is true

$$f(P_1 x_1 + (1 - P_1) x_2) \leq P_1 f(x_1) + (1 - P_1) f(x_2),$$

which is exactly what we wanted to show.

Now we assume it is true for $n = k$, and show it is true for $n = k + 1$.

$$f\left(\sum_{j=1}^{k} P_j x_j\right) \leq \sum_{j=1}^{k} P_j f(x_j) \text{ is true.}$$

We want to show that

$$f\left(\sum_{j=1}^{k+1} P_j x_j\right) \leq \sum_{j=1}^{k+1} P_j f(x_j) \text{ is true.}$$

We know that

$$\sum_{j=1}^{k+1} P_j x_j = P_{k+1} x_{k+1} + \sum_{j=1}^{k} P_j x_j,$$

which means that

$$f\left(\sum_{j=1}^{k+1} P_j x_j\right) = f\left(P_{k+1} x_{k+1} + \sum_{j=1}^{k} P_j x_j\right).$$

Recall that convexity says that

$$f(px + (1-p)y) \leq pf(x) + (1-p)f(y),$$

so let's multiply and divide the last summation by $(1 - P_{k+1})$, so we have

$$f\left(\sum_{j=1}^{k+1} P_j x_j\right) = f\left(P_{k+1} x_{k+1} + (1 - P_{k+1}) \sum_{j=1}^{k} \frac{P_j x_j}{1 - P_{k+1}}\right) \leq P_{k+1} f(x_{k+1}) + (1 - P_{k+1}) f\left(\sum_{j=1}^{k} \frac{P_j x_j}{1 - P_{k+1}}\right)$$

by our assumption for $n = k$,

$$P_{k+1} f(x_{k+1}) + (1 - P_{k+1}) f\left(\sum_{j=1}^{k} \frac{P_j x_j}{1 - P_{k+1}}\right) \leq P_{k+1} f(x_{k+1}) + (1 - P_{k+1}) \sum_{j=1}^{k} \frac{P_j}{1 - P_{k+1}} f(x_j)$$

$$P_{k+1} f(x_{k+1}) + (1 - P_{k+1}) \sum_{j=1}^{k} \frac{P_j}{1 - P_{k+1}} f(x_j) = P_{k+1} f(x_{k+1}) + \sum_{j=1}^{k} P_j f(x_j) = \sum_{j=1}^{k+1} P_j f(x_j)$$

□

Example 11.12. For a triangle with area A and sides a, b, and c:

$$\max\{ab, ac, bc\} \geq \frac{4}{\sqrt{3}} A$$

As a side note, let's assume $a = b = c := s$, so the triangle is equilateral. In this case the area is

$$A = \frac{1}{2} \cdot s \cdot \sqrt{\frac{3s^2}{4}} = \frac{\sqrt{3}s^2}{4},$$

and so the product of two sides is

$$s^2 = \frac{4}{\sqrt{3}}A,$$

which is exactly what we stated at the beginning. This was a geometric interpretation of this rule, but we can also prove it using Jensen's inequality.

First let's define the angles between the sides:

$$\begin{cases} \alpha := \angle bc \\ \beta := \angle ac \\ \gamma := \angle ab \end{cases}$$

It follows then that the area can be given by

$$A = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ab\sin\gamma,$$

and the products of the angles can then be given by

$$\begin{cases} bc = \dfrac{2A}{\sin\alpha} \\ ac = \dfrac{2A}{\sin\beta} \\ ab = \dfrac{2A}{\sin\gamma} \end{cases}$$

The average of a set of numbers cannot be greater than the maximum of those numbers, so

$$\frac{1}{3}(ab + ac + bc) \leq \max\{ab, ac, bc\}.$$

Now we substitute the products on the left hand side with their definitions involving sine

$$\frac{1}{3}(ab + ac + bc) = \frac{2A}{3}\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\beta}\right) \leq \max\{ab, ac, bc\}.$$

We want to show that

$$\frac{2A}{3}\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\beta}\right)\geq\frac{4}{\sqrt{3}}A.$$

Now, we could define the $f(x)$ in Jensen's inequality to be $1/\sin(x)$. On the interval $[0,\pi]$, $f(x)$ is convex, so it seems ok. We can define $P_i = 1/3$, and $x_i \in \{\alpha,\beta,\gamma\}$. We also know that $\alpha+\beta+\gamma=\pi$, so

$$\frac{1}{\sin\left(\sum_{x=\alpha,\beta,\gamma}\frac{1}{3}\right)}\leq\sum_{x=\alpha,\beta,\gamma}\frac{1}{3}\frac{1}{\sin(x)}=\frac{1}{3}\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\beta}\right)$$

$$\frac{1}{\sin\left(\sum_{x=\alpha,\beta,\gamma}\frac{1}{3}\right)}=\frac{1}{\sin\left(\frac{1}{3}(\alpha+\beta+\gamma)\right)}=\frac{1}{\sin\left(\frac{1}{3}\pi\right)}=\frac{2}{\sqrt{3}}$$

$$\frac{2}{\sqrt{3}}\leq\frac{1}{3}\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\beta}\right)$$

Now multiply both sides by $2A$ and we get exactly what we wanted to show:

$$\frac{2A}{3}\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\beta}\right)\geq\frac{4}{\sqrt{3}}A.$$

Example 11.13. Show that if $x, z, z > 0$, and $x + y + z = 1$, then

$$64\leq\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right).$$

Note that we have a product on the right hand side, but Jensen's inequality applies to summations. Logarithms have the property of turning products into summations, so

$$f(x)=\ln\left[1+\frac{1}{x}\right]$$

Example 11.14.

$$\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1}>\frac{3}{x}\ \forall\ x>1$$