Griffiths Chapter 1

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Problem 1.3

Consider the gaussian distribution

$$\rho(x) = A \exp(-\lambda(x-a)^2),$$

where A, a, and λ are positive real constants.

a. Use Equation 1.16 to determine A.

$$1 = \int_{-\infty}^{\infty} A \exp(-\lambda(x-a)^2) \, \mathrm{d}\, x$$

Pull out constant.

$$1 = A \int_{-\infty}^{\infty} \exp(-\lambda(x-a)^2) \, \mathrm{d}\, x$$

Let $u = x - a$, $\mathrm{d}\, u = \mathrm{d}\, x$

$$1 = A \int_{-\infty}^{\infty} \exp(-\lambda u^2) \, \mathrm{d}\, u$$

From integral table:

$$\int_{-\infty}^{\infty} e^{-cx^2} \, \mathrm{d}\, x = \sqrt{\frac{\pi}{c}}$$

Substitute:

$$1 = A\sqrt{\frac{\pi}{\lambda}} \implies A = \sqrt{\frac{\lambda}{\pi}}$$

Now we have

$$\rho(x) = \sqrt{\frac{\pi}{\lambda}} \exp(-\lambda(x-a)^2)$$

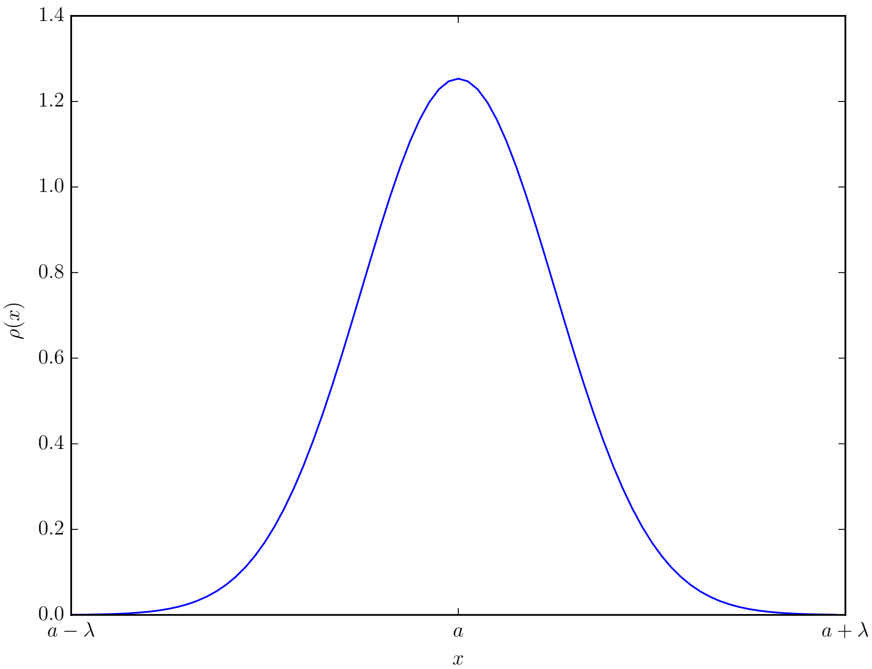
b. Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x) \, \mathrm{d}\, x = \int_{-\infty}^{\infty} x\sqrt{\frac{\pi}{\lambda}} \exp(-\lambda(x-a)^2) = \frac{\pi a}{\lambda}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x) \, \mathrm{d}\, x = \int_{-\infty}^{\infty} x^2\sqrt{\frac{\pi}{\lambda}} \exp(-\lambda(x-a)^2) = \frac{\pi a^2}{\lambda} + \frac{\pi}{2\lambda^2}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\pi a^2}{\lambda} + \frac{\pi}{2\lambda^2} - \left(\frac{\pi a}{\lambda}\right)^2} = \sqrt{\frac{\pi\left(2a^2\lambda - 2\pi a^2 + 1\right)}{2\lambda^2}}$$

c. Sketch the graph of $\rho(x)$.



Problem 1.4

At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A\dfrac{x}{a}, & \text{if } 0 \leq x \leq a, \\ A\dfrac{b-x}{b-a}, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are constants.

a. Normalize Ψ (that is, find A, in terms of a and b).

$$\rho(x,0) = |\Psi(x,0)|^2 = \Psi^2(x,0) = A^2 \begin{cases} \dfrac{x^2}{a^2}, & \text{if } 0 \le x \le a, \\ \dfrac{(b-x)^2}{(b-a)^2}, & \text{if } a \le x \le b, \\ 0, & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{+\infty} \rho(x)\,\mathrm{d}\,x = A^2 \int_0^a \frac{x^2}{a^2}\,\mathrm{d}\,x + A^2 \int_a^b \frac{(b-x)^2}{(b-a)^2}\,\mathrm{d}\,x$$

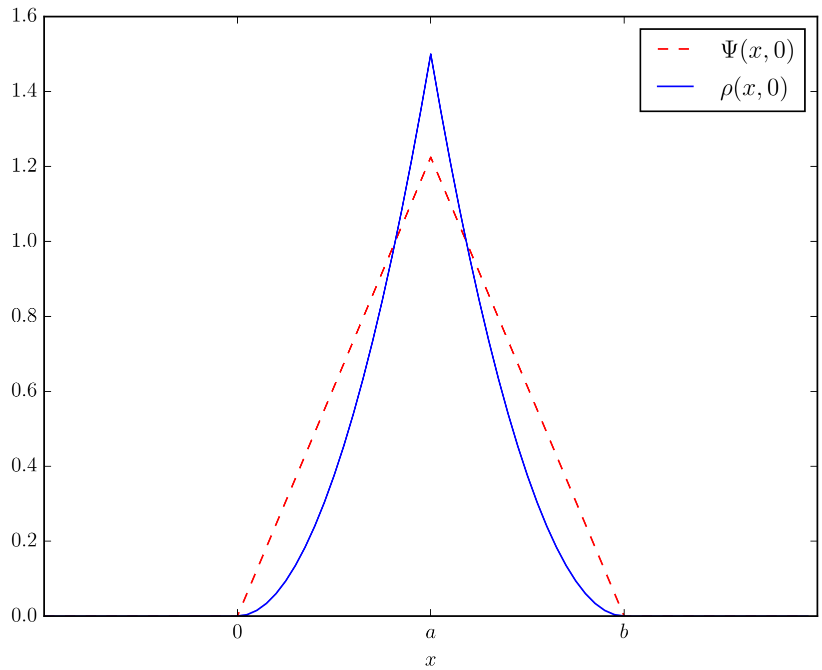
$$u := b - x \implies \mathrm{d}\,u = -\,\mathrm{d}\,x; \quad x = a \implies u = b - a; \quad x = b \implies u = 0$$

$$A^{-2} = \int_0^a \frac{x^2}{a^2}\,\mathrm{d}\,x - \int_{b-a}^0 \frac{u^2}{(b-a)^2}\,\mathrm{d}\,u = \int_0^a \frac{x^2}{a^2}\,\mathrm{d}\,x + \int_0^{b-a} \frac{u^2}{(b-a)^2}\,\mathrm{d}\,u$$

$$= \frac{1}{3}\left[\frac{x^3}{a^2}\right]_0^a + \frac{1}{3}\left[\frac{u^3}{(b-a)^2}\right]_0^{b-a} = \frac{1}{3}\left[a + (b-a)\right] = \frac{b}{3} \implies A = \sqrt{\frac{3}{b}}$$

$$\Psi(x,0) = \sqrt{\frac{3}{b}} \begin{cases} \dfrac{x}{a}, & \text{if } 0 \le x \le a, \\ \dfrac{b-x}{b-a}, & \text{if } a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

b. Sketch $\Psi(x,0)$ as a function of x.



c. Where is the particle most likely to be found, at $t = 0$?

The particle's most likely position is given by $\underset{x}{\operatorname{argmax}}\ \rho(x,0)$. To the left of a, ρ is positive and increasing, to the right of it, it is positive and decreasing, and outside the interval $[0,b]$, it is zero, therefore the most likely position is at $x = a$.

d. What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

$$P[x < a] = \frac{3}{b}\int_0^a \frac{x^2}{a^2}\,\mathrm{d}\,x = \frac{3}{b}\cdot\frac{1}{3}\left[\frac{x^3}{a^2}\right]_0^a = \frac{a}{b}$$

In the limiting case of $b = a$, this gives a probability of 1, which is to be expected as the probability is 1 over the interval $[0, b]$, which is now the same as $[0, a]$. In the limiting case of $b = 2a$, the probability is 1/2, which is also expected, as $P(x, 0)$ is symmetric about a when both intervals have equal size, distributing half of the probability on $[0, a]$ and half on $[a, b]$.

e. What is the expectation value of x?

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}\,x = \frac{3}{b}\left[\int_0^a \frac{x^3}{a^2}\,\mathrm{d}\,x + \int_a^b x\frac{(b-x)^2}{(b-a)^2}\,\mathrm{d}\,x\right]\\ &= \frac{3}{b}\left[\frac{b(2a+b)}{12}\right] = \frac{2a+b}{4}\end{aligned}$$

Problem 1.5

Consider the wave function

$$\Psi(x, t) = A\exp(-\lambda|x|)\exp(-\imath\omega t),$$

where A, λ, and ω are positive real constants.

a. Normalize Ψ

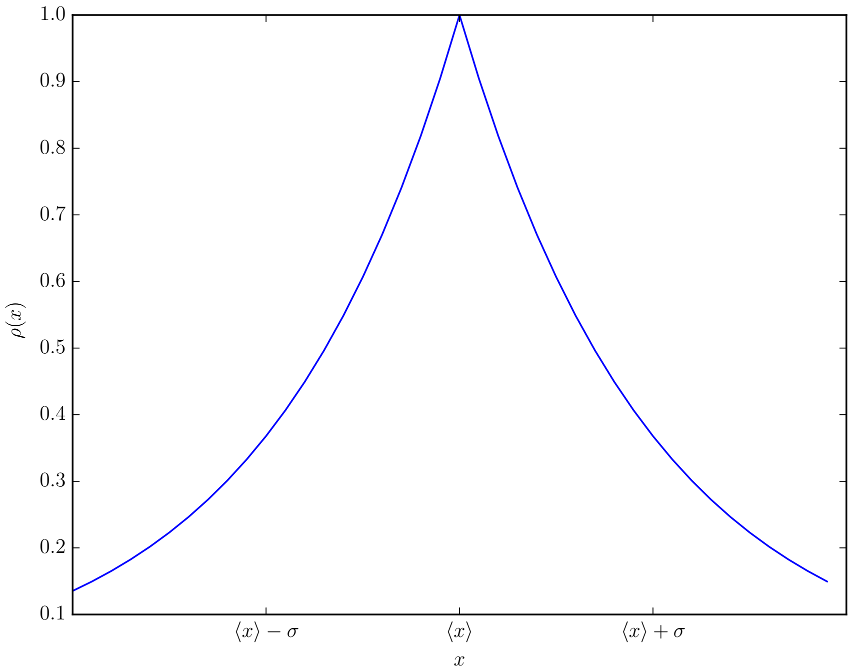
$$\begin{aligned}\rho(x) &= |\Psi(x,t)|^2 = \Psi^*\Psi\\ \Psi &= A\exp(-\lambda|x| - \imath\omega t)\\ \Psi^* &= A\exp(-\lambda|x| + \imath\omega t)\\ \rho(x) &= A^2\exp(-2\lambda|x|)\\ 1 &= \int_{-\infty}^{+\infty}\rho(x)\,\mathrm{d}\,x\\ A^{-2} &= \int_{-\infty}^{+\infty}\exp(-2\lambda|x|)\,\mathrm{d}\,x = \frac{1}{\lambda}\\ A &= \sqrt{\lambda}\\ \rho(x) &= \lambda\exp(-2\lambda|x|)\\ \Psi(x,t) &= \sqrt{\lambda}\exp(-\lambda|x| - \imath\omega t)\end{aligned}$$

b. Determine the expectation values of x and x^2.

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{+\infty} x\rho(x)\,\mathrm{d}\,x = \int_{-\infty}^{+\infty} x\lambda\exp(-2\lambda|x|) = 0\\ \langle x^2\rangle &= \int_{-\infty}^{+\infty} x^2\rho(x)\,\mathrm{d}\,x = \int_{-\infty}^{+\infty} x^2\lambda\exp(-2\lambda|x|) = \frac{1}{2\lambda^2}\end{aligned}$$

c. Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle + \sigma)$ and $(\langle x\rangle - \sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

$$\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle - 0} = \langle x^2\rangle = \frac{1}{2\lambda^2}$$



The probability of finding the particle *outside* this range is the complement of the probability of finding the particle *inside* this range, which is given by

$$\begin{aligned}\rho[\langle x\rangle - \sigma < x < \langle x\rangle + \sigma]' &= 1 - \rho[\langle x\rangle - \sigma < x < \langle x\rangle + \sigma] \\ &= 1 - \int_{\langle x\rangle - \sigma}^{\langle x\rangle + \sigma} \rho(x)\, \mathrm{d}\, x \\ &= 1 - \int_{-(2\lambda^2)^{-1}}^{(2\lambda^2)^{-1}} \lambda \exp(-2\lambda|x|)\, \mathrm{d}\, x \\ &= 1 - \big(1 - \exp(-\lambda^{-1})\big) = \exp(-\lambda^{-1})\end{aligned}$$

Problem 1.7

Calculate $\mathrm{d}\,\langle p\rangle \,/\, \mathrm{d}\, t$.

$$\begin{aligned}\langle p\rangle &= -\imath\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, \mathrm{d}\, x \\ \frac{\mathrm{d}\,\langle p\rangle}{\mathrm{d}\, t} &= -\imath\hbar \frac{\mathrm{d}}{\mathrm{d}\, t} \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x}\, \mathrm{d}\, x = -\imath\hbar \int_{-\infty}^{\infty} \left[\Psi^* \frac{\partial}{\partial t}\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x}\right] \mathrm{d}\, x\end{aligned}$$

From Schrödinger equation:

$$\frac{\partial \Psi}{\partial t} = \frac{\imath\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\imath}{\hbar}V\Psi; \quad \frac{\partial \Psi^*}{\partial t} = -\frac{\imath\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{\imath}{\hbar}V\Psi^*$$

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}\,t} = -\imath\hbar\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial}{\partial t}\frac{\partial \Psi}{\partial x} + \left(-\frac{\imath\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\imath}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x}\right]\mathrm{d}\,x$$

Note that the wavefunction has continuous second partial derivatives, and therefore the partial derivatives are *commutative* (by Schwarz' theorem)

$$\begin{aligned}
\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}\,t} &= -\imath\hbar\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial}{\partial x}\frac{\partial \Psi}{\partial t} + \left(-\frac{\imath\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\imath}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x}\right]\mathrm{d}\,x \\
&= -\imath\hbar\int_{-\infty}^{\infty}\left[\Psi^*\frac{\partial}{\partial x}\left(\frac{\imath\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{\imath}{\hbar}V\Psi\right) + \left(-\frac{\imath\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\imath}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x}\right]\mathrm{d}\,x \\
&= -\imath\hbar\int_{-\infty}^{\infty}\left\{\Psi^*\left[\frac{\imath\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \frac{\imath}{\hbar}\left(\frac{\partial V}{\partial x}\Psi + V\frac{\partial \Psi}{\partial x}\right)\right] + \left(-\frac{\imath\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\imath}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x}\right\}\mathrm{d}\,x \\
&= -\imath\hbar\int_{-\infty}^{\infty}\left\{\frac{\partial \Psi}{\partial x}\left[\Psi^*\left(-\frac{\imath}{\hbar}V + \frac{\imath}{\hbar}V\right) - \frac{\imath\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\right] + \Psi^*\left[\frac{\imath\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \frac{\imath}{\hbar}\frac{\partial V}{\partial x}\Psi\right]\right\}\mathrm{d}\,x \\
&= -\imath\hbar\int_{-\infty}^{\infty}\left\{-\frac{\imath\hbar}{2m}\frac{\partial \Psi}{\partial x}\frac{\partial^2\Psi^*}{\partial x^2} + \Psi^*\left[\frac{\imath\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \frac{\imath}{\hbar}\frac{\partial V}{\partial x}\Psi\right]\right\}\mathrm{d}\,x \\
&= -\imath\hbar\int_{-\infty}^{\infty}\Psi^*\left(-\frac{\imath}{\hbar}\frac{\partial V}{\partial x}\right)\Psi\,\mathrm{d}\,x - \imath\hbar\int_{-\infty}^{\infty}\left(\Psi^*\frac{\imath\hbar}{2m}\frac{\partial^3\Psi}{\partial x^3} - \frac{\imath\hbar}{2m}\frac{\partial \Psi}{\partial x}\frac{\partial^2\Psi^*}{\partial x^2}\right)\mathrm{d}\,x \\
&= \int_{-\infty}^{\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi\,\mathrm{d}\,x + \frac{\hbar^2}{2m}\left[\int_{-\infty}^{\infty}\Psi^*\frac{\partial^3\Psi}{\partial x^3}\,\mathrm{d}\,x - \int_{-\infty}^{\infty}\frac{\partial \Psi}{\partial x}\frac{\partial^2\Psi^*}{\partial x^2}\,\mathrm{d}\,x\right]
\end{aligned}$$

Let $u = \Psi^*$, then $\mathrm{d}\,u = \frac{\partial \Psi^*}{\partial x}\,\mathrm{d}\,x$. Let $\mathrm{d}\,v = \frac{\partial^3\Psi}{\partial x^3}\,\mathrm{d}\,x$, then $v = \frac{\partial^2\Psi}{\partial x^2}$.

Let $\mu = \frac{\partial \Psi}{\partial x}$, then $\mathrm{d}\,\mu = \frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}\,x$. Let $\mathrm{d}\,\nu = \frac{\partial^2\Psi^*}{\partial x^2}\,\mathrm{d}\,x$, then $\nu = \frac{\partial \Psi^*}{\partial x}$.

Using integration by parts, that gives:

$$\begin{aligned}
\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}\,t} &= \int_{-\infty}^{\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi\,\mathrm{d}\,x + \frac{\hbar^2}{2m}\left\{\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial^2\Psi}{\partial x^2}\frac{\partial \Psi^*}{\partial x}\,\mathrm{d}\,x - \left[\left(\frac{\partial \Psi}{\partial x}\frac{\partial \Psi^*}{\partial x}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial \Psi^*}{\partial x}\frac{\partial^2\Psi}{\partial x^2}\,\mathrm{d}\,x\right]\right\} \\
&= \int_{-\infty}^{\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi\,\mathrm{d}\,x + \frac{\hbar^2}{2m}\left[\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}\right)_{-\infty}^{\infty} - \left(\frac{\partial \Psi}{\partial x}\frac{\partial \Psi^*}{\partial x}\right)_{-\infty}^{\infty}\right]
\end{aligned}$$

$\Psi(x,t)$ must go to zero as x goes to $(\pm)$ infinity, so the entire $\hbar^2/2m$ term is zero.

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}\,t} = \int_{-\infty}^{\infty}\Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi\,\mathrm{d}\,x = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

We have now arrived at Equation 1.38.