Griffiths Chapter 4 (1)

Dan Wysocki

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Problem 4.1

a. Work out all of the **canonical commutation relations** for components of the operators **r** and **p**.

$$\begin{aligned}
[r_i, r_j] &= r_i r_j - r_j r_i = r_i r_j - r_i r_j = 0 \\
[p_i, p_j] &= p_i p_j - p_j p_i = -\hbar^2 \left[\frac{\mathrm{d}}{\mathrm{d}r_i} \frac{\mathrm{d}}{\mathrm{d}r_j} - \frac{\mathrm{d}}{\mathrm{d}r_j} \frac{\mathrm{d}}{\mathrm{d}r_i} \right] \\
&= -\hbar^2 \left[\frac{\mathrm{d}}{\mathrm{d}r_i} \frac{\mathrm{d}}{\mathrm{d}r_j} - \frac{\mathrm{d}}{\mathrm{d}r_i} \frac{\mathrm{d}}{\mathrm{d}r_j} \right] = 0 \\
[r_i, p_j]\, f &= r_i p_j(f) - p_j r_i(f) = r_i \frac{\hbar}{\imath} \frac{\mathrm{d}}{\mathrm{d}r_j} f - \frac{\hbar}{\imath} \frac{\mathrm{d}}{\mathrm{d}r_j} (r_i \cdot f) \\
&= \frac{\hbar}{\imath} \left[r_i \frac{\mathrm{d}f}{\mathrm{d}r_j} - \frac{\mathrm{d}r_i}{\mathrm{d}r_j} f - r_i \frac{\mathrm{d}f}{\mathrm{d}r_j} \right] = \frac{\hbar}{\imath} \left[- \frac{\mathrm{d}r_i}{\mathrm{d}r_j} f \right] \\
\implies [r_i, p_j] &= \imath\hbar \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} = \imath\hbar\delta_{ij}
\end{aligned}$$

b. Confirm Ehrenfest's theorem for 3-dimensions

Ehrenfest's theorem is simply a special case of Equation 3.71, so we make use of that with $Q = r_i$ and $\partial Q / \partial t = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle r_i \rangle = \frac{\imath}{\hbar} \langle [H, r_i] \rangle .$$

First we find $[H, r_i]$:

$$[H, r_i] = \left(\frac{p^2}{2m} + V \right) r_i - r_i \left(\frac{p^2}{2m} + V \right) = \frac{p^2}{2m} r_i - r_i \frac{p^2}{2m} = \frac{1}{2m} [p^2, r_i] ,$$

since we know that $[r_i, p_j] = \imath\hbar\delta_{ij}$, the components of p aside from p_i disappear and we're left with

$$\begin{aligned}
[H, r_i]\, f &= \frac{1}{2m} [p_i^2, r_i]\, f = -\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}^2}{\mathrm{d}r_i^2} (r_i f) - r_i \frac{\mathrm{d}^2}{\mathrm{d}r_i^2} f \right) = -\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}}{\mathrm{d}r_i} \left(\frac{\mathrm{d}r_i}{\mathrm{d}r_i} f + r_i \frac{\mathrm{d}f}{\mathrm{d}r_i} \right) - r_i \frac{\mathrm{d}^2 f}{\mathrm{d}r_i^2} \right) \\
&= -\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}}{\mathrm{d}r_i} \left(f + r_i \frac{\mathrm{d}f}{\mathrm{d}r_i} \right) - r_i \frac{\mathrm{d}^2 f}{\mathrm{d}r_i^2} \right) = -\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}f}{\mathrm{d}r_i} + \frac{\mathrm{d}r_i}{\mathrm{d}r_i} \frac{\mathrm{d}f}{\mathrm{d}r_i} + r_i \frac{\mathrm{d}^2 f}{\mathrm{d}r_i^2} - r_i \frac{\mathrm{d}^2 f}{\mathrm{d}r_i^2} \right) \\
&= -\frac{\hbar^2}{2m} \left(\frac{\mathrm{d}f}{\mathrm{d}r_i} + \frac{\mathrm{d}f}{\mathrm{d}r_i} \right) = -\frac{\hbar^2}{m} \frac{\mathrm{d}f}{\mathrm{d}r_i} \implies [H, r_i] = -\frac{\hbar^2}{m} \frac{\mathrm{d}}{\mathrm{d}r_i} = -\frac{\imath\hbar}{m} p_i
\end{aligned}$$

Now we substitute this into Equation 3.71 to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle r_i\rangle = \frac{\imath}{\hbar}\left\langle -\frac{\imath\hbar}{m}p_i\right\rangle = \frac{1}{m}\langle p_i\rangle \implies \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{r}\rangle = \frac{1}{m}\langle \mathbf{p}\rangle.$$

Now repeat the same process, this time with $Q = p_i$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p_i\rangle = \frac{\imath}{\hbar}\langle [H, p_i]\rangle.$$

$$[H, p_i] = \left(\frac{p^2}{2m} + V\right)p_i - p_i\left(\frac{p^2}{2m} + V\right) = \frac{p_i^3}{2m} + Vp_i - \frac{p_i^3}{2m} + p_iV = Vp_i - p_iV = [V, p_i]$$

By Equation 3.65, we know that $[f(x), p] = \imath\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$, which can be extended to $[f(x), \mathbf{p}] = \imath\hbar\boldsymbol{\nabla}f$. So we have $[H, p_i] = [V, p_i] = \imath\hbar\boldsymbol{\nabla}V$. Substituting into Equation 3.71 gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{p}\rangle = \frac{\imath}{\hbar}\langle \imath\hbar\boldsymbol{\nabla}V\rangle = \langle -\boldsymbol{\nabla}V\rangle.$$

c. Formulate Heisenberg's uncertainty principle in three dimensions.

Using the generalized uncertainty principle

$$\sigma_A^2\sigma_B^2 \geqslant \left(\frac{1}{2\imath}\langle [A, B]\rangle\right)^2$$

substituting the commutation relations for r_i and p_j above, it is clear that

$$\sigma_{r_i}^2\sigma_{p_j}^2 \geqslant \frac{\hbar^2}{4}\delta_{ij}, \text{ or } \sigma_{r_i}\sigma_{p_j} \geqslant \frac{\hbar}{2}\delta_{ij}$$

Problem 4.2 Use separation of variables in *cartesian* coordinates to solve the infinite *cubical* well (or "particle in a box"):

$$V(x, y, z) = \begin{cases} 0, & \text{if } x,\ y,\ z \text{ are all between 0 and } a; \\ \infty & \text{otherwise.} \end{cases}$$

a. Find the stationary states, and the corresponding energies.

The time-independent Schrödinger equation in 3D states that

$$E\psi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}),$$

where $\mathbf{r} = [\,x\ y\ z\,]$, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Now let $\psi(\mathbf{r}) = X(x)Y(y)Z(z)$. $\nabla^2\psi = X''YZ + XY''Z + XYZ''$. Inside the box, $V = 0$, so the Schrödinger equation simplifies to

$$EXYZ = -\frac{\hbar^2}{2m}X''YZ + XY''Z + XYZ'',$$

and dividing by XYZ and $-\hbar^2/2m$ we get

$$-\frac{2mE}{\hbar^2} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}.$$

Each term in the summation on the right is a function of a different variable, yet they sum to a constant. This can only be the case if each term is in fact constant itself. We shall call these constants $-\lambda_x^2$, $-\lambda_y^2$, and $-\lambda_z^2$. We make these negative in order to make the energy positive, and we square them for reasons that will soon be made clear. Substituting and solving for E gives:

$$E = \frac{\hbar^2}{2m}\left(\lambda_x^2 + \lambda_y^2 + \lambda_z^2\right),$$

and separating the three variables into a system of equations gives:

$$X'' + \lambda_x^2 X = 0; \quad Y'' + \lambda_y^2 Y = 0; \quad Z'' + \lambda_z^2 Z = 0.$$

Now we will solve the ODE for $X(x)$, and in doing so will solve the ODE's for $Y(y)$ and $Z(z)$ (just change the variable and constant names). The ODE for $X(x)$ is a familiar ODE, with solution

$$X(x) = A_x \sin(\lambda_x x) + B_x \cos(\lambda_x x).$$

Note that we now have λ_x instead of λ_x^2, but we could have, with equal validity, originally chosen constants μ_x, which would leave us now with $\sin\left(\sqrt{\mu_x} x\right)$.

Now consider the boundary conditions. As we know the potential is infinite outside the box, X must go to zero at the boundaries. So first we take the boundary condition $X(0) = 0$:

$$X(0) = A_x \sin(0) + B_x \cos(0) = B_x = 0.$$

This removes the B_x term from the equation, and we can now take the second boundary condition, $X(a) = 0$:

$$X(a) = A_x \sin(\lambda_x a) = 0.$$

This can imply either of two cases:

1. $A_x = 0$, in which case $X \equiv 0$, and therefore $\psi \equiv 0$, which is non-normalizable, so we disregard it.
2. $\sin(\lambda_x a) = 0$, in which case $\lambda_x = n_x \pi / a$ for any non-zero integer n_x. Negative values of n are redundant, and so we impose the constraint $n_x \in \mathbb{Z}^+$.

Now we can rewrite X as

$$X_n(x) = A_x \sin\left(\frac{n_x \pi x}{a}\right),$$

and normalize over $[0, a]$

$$\int_0^a X_n^* X_n \,\mathrm{d}x = |A_x|^2 \int_0^a \sin^2\left(\frac{n_x \pi x}{a}\right) \mathrm{d}x = |A_x|^2 \frac{a}{2} = 1 \implies A_x = \sqrt{2/a}.$$

Note that A_x is independent of x and n_x, and so $A_x = A_y = A_z$.

So we have now shown that

$$X_n(x) = \sqrt{2/a}\sin\left(\frac{n_x \pi x}{a}\right); \quad Y_n(y) = \sqrt{2/a}\sin\left(\frac{n_y \pi y}{a}\right); \quad Z_n(z) = \sqrt{2/a}\sin\left(\frac{n_z \pi z}{a}\right),$$

$$\psi_n(\mathbf{r}) = A_x A_y A_z \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) = (2/a)^{3/2} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right),$$

$$E_n = \frac{\pi^2 \hbar^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right).$$

b. Call the distinct energies E_1, E_2, E_3, ..., in order of increasing energy. Find E_1 through E_6. Determine their degeneracies (that is, the number of different states that share the same energy).

As there will be many states with the same energy, and computing by hand will be tedious, I make use of the following `python` script to output LaTeX source for the following table.

```python
#!/usr/bin/python

from collections import defaultdict
from itertools import islice, product

def energy_states(highest_state):
    # a mapping of
    # E -> [(n_x, n_y, n_z), ...]
    states = defaultdict(list)

    for x, y, z in product(range(1, highest_state+1), repeat=3):
        E = x**2+y**2+z**2
        states[E].append([x, y, z])

    return states

def display(states, n_low, n_high):
    print(r"\begin{tabular}{ccc||r}")
    print(r"$n_x$ & $n_y$ & $n_z$ & $(n_x^2 + n_y^2 + n_z^2)$ \\\hline")

    # display the energy states from n_low to n_high
    for E in islice(sorted(states), n_low, n_high):
        print(r"\hline")
        for x, y, z in states[E]:
            print(r"{} & {} & {} & {}".format(x, y, z, E))
            print(r"\\")

    print(r"\end{tabular}")

if __name__ == "__main__":
    states = energy_states(5)

    print("Part b")
    display(states, 0, 6)

    print("Part c")
    # start from 13 since it is indexed by 0
    # stop at 14 because the end of islice is exclusive
    display(states, 13, 14)
```

n_x	n_y	n_z	$(n_x^2 + n_y^2 + n_z^2)$
1	1	1	3
1	1	2	6
1	2	1	6
2	1	1	6
1	2	2	9
2	1	2	9
2	2	1	9
1	1	3	11
1	3	1	11
3	1	1	11
2	2	2	12
1	2	3	14
1	3	2	14
2	1	3	14
2	3	1	14
3	1	2	14
3	2	1	14

$$E_1 = \frac{3\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 1}$$
$$E_2 = \frac{6\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 3}$$
$$E_3 = \frac{9\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 3}$$
$$E_4 = \frac{11\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 3}$$
$$E_5 = \frac{12\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 1}$$
$$E_6 = \frac{14\pi^2\hbar^2}{2ma^2}; \qquad \text{degeneracy is 6}$$

c. What is the degeneracy of E_{14} and why is this case interesting?

The following table displays the combinations of n which form E_{14}. There are 4 combinations, so the degeneracy is 4.

n_x	n_y	n_z	$(n_x^2 + n_y^2 + n_z^2)$
1	1	5	27
1	5	1	27
3	3	3	27
5	1	1	27

Problem 4.16 A **hydrogenic atom** consists of a single electron orbiting a nucleus with Z protons ($Z = 1$ would be hydrogen itself, $Z = 2$ is ionized helium, $Z = 3$ is doubly ionized lithium, and so on). Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a(Z)$, and the Rydberg constant $R(Z)$ for a hydrogenic atom. (Express your answers as appropriate multiples of the hydrogen values.) Where in the electromagnetic spectrum would the Lyman series fall, for $Z = 2$ and $Z = 3$?

The Coloumb potential is given by

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r},$$

and for a hydrogenic atom with Z protons, $Q = Ze$ and $q = -e$, so

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}.$$

We now substitute $V(r)$ into the radial wave equation, giving us

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left[-Z\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

Now we define

$$\kappa = \frac{\sqrt{-2mE}}{\hbar},$$

and divide through by E, giving

$$\frac{1}{\kappa^2}\frac{\mathrm{d}^2u}{\mathrm{d}r^2} = \left[1 - Z\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{\ell(\ell+1)}{(\kappa r)^2}\right]u.$$

Now we define

$$\rho \equiv \kappa r, \quad \text{and} \quad \rho_0 \equiv \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa},$$

giving us

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[1 - \frac{Z\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u.$$

Here is where the trick comes in. Let us define

$$\tilde{\rho}_0 \equiv Z\rho_0.$$

Now the wave equation is given by

$$\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} = \left[1 - \frac{\tilde{\rho}_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u.$$

If we treat $\tilde{\rho}_0$ as we treated ρ_0 when dealing with the hydrogen atom, we wind up with the same results, up until the very end, where

$$\tilde{\rho}_0 = 2n.$$

Unravelling our definition of $\tilde{\rho}_0$ gives us

$$\tilde{\rho}_0 = Z\frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} = Z\frac{me^2}{2\pi\varepsilon_0\hbar\sqrt{-2mE}}.$$

Now we solve for $E_n(Z)$

$$2n = Z\frac{me^2}{2\pi\varepsilon_0\hbar\sqrt{-2mE}} \implies \sqrt{-2mE} = Z\frac{me^2}{4\pi\varepsilon_0\hbar n} \implies E_n(Z) = -\frac{m}{2\hbar^2}\left[Z\frac{e^2}{4\pi\varepsilon_0}\right]^2\frac{1}{n^2}.$$

Now Equation 4.70 gives E_n for hydrogen, which I will denote $E_{H,n}$, as

$$E_{H,n} = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\frac{1}{n^2},$$

and so E_n for a hydrogenic atom is simply

$$E_n(Z) = Z^2E_{H,n}.$$

The binding energy is therefore $E_1(Z)$

$$E_1(Z) = Z^2 E_{H,1} = -Z^2 \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = -Z^2 13.6\,\mathrm{eV}.$$

To find the Bohr radius, we note that

$$\kappa = Z\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right)\frac{1}{n} = Z\frac{1}{a_H n} = \frac{1}{a(Z)n},$$

where a_H is the Bohr radius of hydrogen. It follows that $a(Z) = a_H/Z$, and so

$$a(Z) = \frac{1}{Z}\frac{4\pi\varepsilon_0\hbar^2}{me^2} = \frac{0.529\times10^{-10}\,\mathrm{m}}{Z}$$

Now, to find the Rydberg constant, $R(Z)$, we first look at the energy difference E_γ

$$E_\gamma(Z) = E_i(Z) - E_f(Z) = -Z^2\,13.6\,\mathrm{eV}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right),$$

and then translate that into $1/\lambda$ via $E = hc/\lambda$, giving

$$\frac{1}{\lambda} = \frac{E_\gamma(Z)}{hc} = -Z^2 13.6\,\mathrm{eV}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) = Z^2 R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R(Z)\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right),$$

so $R(Z) = Z^2 R_H = Z^2 1.097\times10^7\,\mathrm{m}^{-1}$.

Since $R(Z) = Z^2 R_H$, it follows that $\lambda(Z)^{-1} = Z^2\lambda_H^{-1}$, or $\lambda(Z) = \lambda_H/Z^2$.

I have taken the liberty of looking up the wavelengths for the Lyman series, and so for $Z = 2$, I simply divide those by 4, and for $Z = 3$, I divide by 9. The following table summarizes the results

Z	1	2	3
λ ($n = 2$)	121.6 nm	30.40 nm	13.51 nm
λ ($n = \infty$)	91.18 nm	22.80 nm	10.13 nm
class	far / extreme UV	extreme UV	extreme UV

(Lyman series wavelengths taken from https://en.wikipedia.org/wiki/Lyman_series; EM Spectrum classes taken from http://unihedron.com/projects/spectrum/downloads/spectrum_20090210.pdf)

Problem 4.17 Consider the earth–sun system as a gravitational analog of the hydrogen atom.

a. What is the potential energy function (replacing Equation 4.52)? (Let m be the mass of the earth, and M the mass of the sun.)

Gravitational potential energy is given by

$$V(r) = -G\frac{Mm}{r}$$

b. What is the "Bohr radius," a_g, for this system? Work out the actual number.

To find the Bohr radius, we first note that the given potential $V(r)$ is very similar to that of the hydrogen atom. We merely have to replace Coulomb's constant with G, and Qq with Mm. The radial wave equation is given by

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \left[-G\frac{Mm}{r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu$$

Once again, we define $\kappa = \sqrt{-2mE}/\hbar$, and divide through by $E = -(\kappa\hbar)^2/2m$

$$\frac{1}{\kappa^2}\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \left[1 - 2G\frac{Mm^2}{\hbar^2\kappa}\frac{1}{\kappa r} + \frac{\ell(\ell+1)}{(\kappa r)^2}\right]u.$$

Now here's the trick, we define ρ and ρ_0 as follows

$$\rho \equiv \kappa r, \quad \text{and} \quad \rho_0 \equiv 2G\frac{Mm^2}{\hbar^2\kappa},$$

which puts us back where we were for the hydrogen atom,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u,$$

except that once we unravel our definition of ρ_0, we will have different constants. From the text, we know that

$$\rho_0 = 2n = 2G\frac{Mm^2}{\hbar^2\kappa} = 2G\frac{Mm^2}{\hbar\sqrt{-2mE}} \implies n = G\frac{Mm^2}{\hbar\sqrt{-2mE}}$$

$$\implies \sqrt{-2mE} = G\frac{Mm^2}{\hbar n} \implies E_n = -\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2}$$

So the Bohr energy is

$$E_n = -\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2}$$

To find the Bohr radius, we return to ρ_0 and solve for κ

$$\rho_0 = 2n = 2G\frac{Mm^2}{\hbar^2\kappa} \implies \kappa = \frac{GMm^2}{\hbar^2}\frac{1}{n}$$

Now since $\kappa = 1/(an)$, that means that

$$\frac{1}{a_g n} = \frac{GMm^2}{\hbar^2}\frac{1}{n} \implies a_g = \frac{\hbar^2}{GMm^2},$$

and substituting $M = M_\odot = 1.9891 \times 10^{30}\,\mathrm{kg}$, $m = M_\oplus = 5.9736 \times 10^{24}\,\mathrm{kg}$, $G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$, and $\hbar = 1.054\,571\,5 \times 10^{-34}\,\mathrm{J\,s}$ (constants taken from *An Introduction to Modern Astrophysics, 2nd Edition* by Carroll and Ostlie), we find

$$\begin{aligned}
a_g &= \frac{\left(1.054\,571\,5 \times 10^{-34}\,\mathrm{J\,s}\right)^2}{\left(6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}\right)\left(1.9891 \times 10^{30}\,\mathrm{kg}\right)\left(5.9736 \times 10^{24}\,\mathrm{kg}\right)^2} \\
&= 2.349 \times 10^{-138}\,\mathrm{J^2s^2\ N^{-1}m^{-2}kg^2kg^{-1}kg^{-2}} = 2.349 \times 10^{-138}\,\mathrm{J^2s^2\ N^{-1}m^{-2}\ kg^{-1}} \\
&= 2.349 \times 10^{-138}\,\mathrm{Js^2m^{-1}kg^{-1}} = 2.349 \times 10^{-138}\,\mathrm{kgm^2s^{-2}s^2m^{-1}kg^{-1}} \\
&= 2.349 \times 10^{-138}\,\mathrm{m}\ \text{(tiny!)}
\end{aligned}$$

c. Write down the gravitational "Bohr formula," and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_0, show that $n = \sqrt{r_0/a_g}$. From this, estimate the quantum number n of the earth.

As was noted previously,

$$E_n = -\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2}.$$

Classically, total energy is given by $E = V + T$, where V is potential energy (we've already found this), and T is kinetic energy. Kinetic energy is given by $mv^2/2$, but we do not know v. To get around this, we take advantage of the fact that this is a circular orbit, and therefore has constant radius and velocity, making it centripetal motion. The net force on the Earth is simply the gravitational force exerted by the Sun, so we equate the gravitational force with the centripetal force

$$m\frac{v^2}{r_0} = G\frac{Mm}{r_0^2} \implies mv^2 = G\frac{Mm}{r_0} \implies \frac{1}{2}mv^2 = G\frac{Mm}{2r_0} = T.$$

Now we substitute E_n, V and T into the total energy equation

$$-\frac{m}{2\hbar^2}(GMm)^2\frac{1}{n^2} = -G\frac{Mm}{r_0} + G\frac{Mm}{2r_0} = -G\frac{Mm}{2r_0}.$$

Solving for n we get

$$\frac{1}{n^2} = G\frac{Mm}{2r_0}\frac{2\hbar^2}{m}\frac{1}{(GMm)^2} = \frac{1}{r_0}\frac{\hbar^2}{GMm^2} = \frac{a_g}{r_0} \implies n = \sqrt{\frac{r_0}{a_g}}$$

We equate r_0 to the semi-major axis of the Earth's orbit, which is 1 AU or $r_0 = 1.495\,978\,706\,6 \times 10^{11}$ m (from Carroll and Ostlie). Substituting this, along with a_g from earlier, gives

$$n = \sqrt{\frac{1.495\,978\,706\,6 \times 10^{11}\text{ m}}{2.349 \times 10^{-138}\text{ m}}} = 2.5236 \times 10^{74}$$

d. Suppose the earth made a transition to the next lower level ($n - 1$). How much energy (in Joules) would be released? What would the wavelength of the emitted photon (or, more likely, graviton) be? (Express your answer in light years – is the remarkable answer a coincidence?)

The energy of the emitted photon would be

$$E_\gamma = E_n - E_{n-1} = -\frac{m}{2\hbar^2}(GMm)^2\left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right).$$

Now, if we were to simply plug in the n from above, and enter it into a calculator, we would get 0 because n is so large that n and $n - 1$ would be identical. We therefore need to find an approximation which contains only a single n term.

$$\frac{1}{n^2} - \frac{1}{(n-1)^2} = \frac{(n-1)^2 - n^2}{n^2(n-1)^2} = \frac{n^2 - 2n + 1 - n^2}{n^2(n-1)^2} = \frac{-2n+1}{n^2(n-1)^2}$$

Here, we make the approximation $n - 1 \approx n$, and $-2n + 1 \approx -2n$, for large n. So

$$\frac{1}{n^2} - \frac{1}{(n-1)^2} \approx \frac{-2n}{n^2 \cdot n^2} = \frac{-2}{n^3},$$

which we then substitute back into E_γ

$$E_\gamma \approx -\frac{m}{2\hbar^2}(GMm)^2\frac{-2}{n^3} = \frac{m}{2\hbar^2}(GMm)^2\frac{2}{n^3},$$

now we plug in the constants

$$E_\gamma = \frac{5.9736 \times 10^{24}\,\mathrm{kg}}{2\left(\cdot 1.054\,571\,5 \times 10^{-34}\,\mathrm{J\,s}\right)^2} \frac{2\left(6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}} \cdot 1.9891 \times 10^{30}\,\mathrm{kg} \cdot 5.9736 \times 10^{24}\,\mathrm{kg}\right)^2}{\left(2.5236 \times 10^{74}\right)^3}$$
$$= 2.0993 \times 10^{-41}\,\mathrm{kgN^2m^4J^{-2}s^{-2}} = 2.0993 \times 10^{-41}\,\mathrm{kgN^2m^4N^{-2}m^{-2}s^{-2}} = 2.0993 \times 10^{-41}\,\mathrm{kgm^2\ s^{-2}}$$
$$= 2.0993 \times 10^{-41}\,\mathrm{J}$$

Now we find wavelength by means of $E = hc/\lambda$, using $h = 6.626\,068 \times 10^{-34}\,\mathrm{J\,s}$ and $c = 2.997\,924\,58 \times 10^8\,\mathrm{ms^{-1}}$ (taken from Carroll and Ostlie),

$$\lambda = \frac{hc}{E_\gamma} = \frac{6.626\,068 \times 10^{-34}\,\mathrm{J\,s} \cdot 2.997\,924\,58 \times 10^8\,\mathrm{ms^{-1}}}{2.0993 \times 10^{-41}\,\mathrm{J}} = 9.4623 \times 10^{15}\,\mathrm{m}.$$

According to Carroll and Ostlie, $1\,\mathrm{ly} = 9.460\,730\,472 \times 10^{15}\,\mathrm{m}$, which is approximately λ, so the wavelength of the emitted graviton is $1\,\mathrm{ly}$.